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Multicriteria decision aid and evolutionary multiobjective optimization with uncertainty management in portfolio optimization

THESIS

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Dedication

To my wife

My everything

To my daughter

My happiness

To my son

My fulfillment

To my parents and brother

My life's story

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List of Acronyms

$\alpha_j(x, y)$ — Marginal likelihood index of the assertion "x is at least as good as y", page 43

$\beta(x, y)$ — likelihood index of the interval-based outranking relation, page 44

β_0 — likelihood threshold for the interval-based outranking method, page 44

δ — likelihood that all criteria in the concordance coalition are in agreement with xSy, page 43

λ — Threshold for a sufficient strength of the concordance coalition, page 42

λ^j — Weight vector of the k criteria, page 52

$\mathbb{E}(R(x))$ — Expected return of portfolio x, page 14

$\mathbb{P}(\omega)$ — Probability of event ω, page 21

$\mathscr{A}$ — Set of portfolios (decision objects, potential actions, alternatives of solution), page 47

$\mathscr{F}$ — Coherent family of criteria (see [46]), page 44

$\mathscr{P}$ — Set of parameters of the interval-based outranking decision model, page 88

μ — Truth degree of a fuzzy predicate, page 105

Ω — Set of feasible alternative solutions, page 13

ψ — Risk aversion indicator, page 15

ρ — Risk measure, page 20

ρ_{ij} — Coefficient of correlation of the returns of the objects x_i and x_j, page 14

Σ — Matrix $n \times n$ of co-variance of the returns, page 14

$\sigma^2(R(x))$ — Variance of portfolio x, page 13

σ_i — Standard deviation of the ith investment object's return, page 14

$\sim$ — Indifference, page 38

$\succ$	Strict Preference, page 38
τ	Represents "an important majority" in the quality index definition of the fundamental analysis, page 108
$\mathbf{C}$	Set of preferentially ordered categories, page 88
$\wedge$	Conjunction operator, page 30
ζ	Vector of expected returns of the n investment objects within the portfolio, page 15
$c(x, y)$	Concordance index of the statement "x is at least as good as y", page 43
$C(xS_\delta y)$	Concordance coalition of the assertion xSy, page 43
$C_{\mathscr{P}}(x)$	Category to which $x \in \mathscr{A}$ is assigned on the basis of $\mathscr{P}$, page 88
$CVaR$	Conditional Value at Risk, page 25
D	Proportion of indicators that reach level n_1 in the quality index definition of the fundamental analysis, page 108
$D(xS_\delta y)$	Discordance coalition of the assertion xSy, page 43
F_x	Distribution function of portfolio x, page 21
$g(\cdot)$	Criteria vector composed of k criteria functions, page 47
H	Piecewise linear function used in the quality index definition of the fundamental analysis, page 108
H_I	Set $\{(x, y) : xI_{\mathscr{P}'}y$ with $C_{\mathscr{P}}(x) \neq C_{\mathscr{P}}(y)\}$, page 89
H_P	Set $\{(x, y) : xP_{\mathscr{P}'}y$ with $C_{\mathscr{P}}(x) \not\succ C_{\mathscr{P}}(y)\}$, page 89
H_Q	Set $\{(x, y) : xQ_{\mathscr{P}'}y \vee xK_{\mathscr{P}'}y$ with $C_{\mathscr{P}}(x) \not\succeq C_{\mathscr{P}}(y)\}$, page 89
if_i	ith fundamental indicator, page 29
it_j^i	Evaluation of the ith stock in the portfolio through the jth technical indicator, page 30
k	Number of objective (criteria) functions, page 47
M	Number of objectives established by the decision maker, page 4
m	Number of inequality constraints, page 47
N	Number of single objective optimization sub-problems used by MOEA/D, page 53
n	Number of decision variables, page 14
n_1	Represents "a sufficiently high level" in the quality index definition of

	the fundamental analysis, page 108
n_2	Threshold for a stock to be considered as *good*, page 108
n_{veto}	Veto value for a stock to be considered as *good*, page 108
p	Number of equality constraints, page 47
$p(I \geq J)$	Possibility function specifying the credibility of I being equal to or greater than J, page 34
p_t^i	Price of the ith stock in the tth period of time, page 30
r	Vector of returns of a portfolio, page 13
$R(x)$	Return of the portfolio x, page 13
r_j^{t+1}	Return of the jth object of a portfolio in period of time $t+1$, page 13
S	Semi-variance, page 23
T	Set of portfolios described by the set of criteria $\mathscr{F}$ and assigned by the DM each to one category of $\mathbf{C}$, page 88
TN	Number of technical indicators considered, page 109
v_j	Veto threshold of the jth criterion, page 42
VaR	Value at Risk, page 24
vol_t^i	Volume (number of shares traded) of the ith stock in period t., page 31
w_j	Weight of the jth criterion, page 42
x	Portfolio; vector solution consisting of n decision variables: $x = [x_1, x_2, \cdots, x_n]^\top$, page 13
x_i	Proportion of resources allocated to the ith investment object, page 13
$xD(\alpha)y$	Denotes that y is α-dominated by x, page 45
xIy	x and y are indifferent, page 45
xKy	x is K preferred to y, page 45
xPy	x is strictly preferred to y, page 45
xQy	x is weakly preferred to y, page 45
xSy	x is at least as good as y, page 43
z^*	Reference point used by MOEA/D, page 52
BB	Bollinger Band, page 30
DC	Double Crossover, page 30
EMA	Exponential Moving Average, page 30

GA	genetic algorithms, page 50
MACD	Moving Average Convergence/Divergence, page 30
MOEA	Multiobjective evolutionary algorithms, page 49
MOEA/D	Multiple Objective Evolutionary Algorithm based on Decomposition, page 52
MPT	Modern Portfolio Theory, page 12
OBV	On Balance Volume, page 30
PF	Pareto front, projection of the set of Pareto-efficient solutions in the criteria space, page 48
R&D	Research and Development, page 19
ROC	Rate of Change, page 30
RSI	Relative Strength Index, page 30
TSI	True Strength Index, page 30
VaR	Value at Risk, page 23

Abstract

A correct distribution of resources among a set of investment objects gives investors a high probability of achieving their objectives. This is very important in modern economy since investors play a crucial role in financial markets around the world. Approaches in charge of recommending a distribution of resources must be able to deal with many conflicting objectives and/or criteria. Furthermore, the uncertainty of the impacts on those objectives and the particular preferences of the investors (including their attitude facing risk) must be considered in order to obtain satisfactory solutions. Finally, the imperfect knowledge in the investors' preferences must be incorporated into the modeling process in order for the approach to be able to effectively reproduce the investors' decisions. The purpose of this doctoral thesis is to propose and validate a comprehensive approach that considers these situations when selecting the distribution of resources maximizing the impact on the objectives of the investor.

This thesis models the imperfect knowledge in the preferences of the investor, as well as uncertainty about the impact on the objectives, using Interval Theory. We propose to characterize the solution alternatives to the Portfolio Optimization Problem using probabilistic confidence intervals to identify the behavior of the investor facing risk. We define an indirect elicitation through an *interval-based Preference Disaggregation Analysis* method in order to obtain an approximation to the real preferences of the investor. Using these preferences the proposed approach then performs a selection pressure towards the most preferred solutions from the investor's perspective in an optimization procedure. We adapt in this thesis the well-known evolutionary multiobjective optimization in order to deal with parameters defined as interval numbers.

The approach has been extensively evaluated using the most common benchmarks with synthetic and real data with in-sample and out-of-sample evaluations. The risk measure and the characterization of portfolios through confidence intervals are evaluated in the context of stock portfolio selection, comparing the performance of our approach with several benchmarks in real historical scenarios. The results indicate that the proposed approach clearly outperforms the benchmarks and that it is able to satisfactorily incorporate the behavior of the investor facing risk. Further experimentation showed that our proposal to build the investor's preferences allows the approach to reproduce the investor's decisions with great effectiveness, in most cases, with more than 99% average effectiveness for portfolios described by up to 12 criteria. A final experimentation showed that using the approximation to the investor preferences allowed the proposed approach to find solutions that are more preferred than reference solutions.

Resumen

Una correcta distribución de recursos entre un conjunto de objetos de inversión brinda a los inversionistas una alta probabilidad de conseguir sus objetivos. Esto es muy importante dado que los inversionistas juegan un papel crucial en los mercados financieros al rededor del mundo. Los métodos encargados de recomendar una distribución de recursos deben ser capaces de considerar muchos objetivos/criterios en conflicto. Además, la incertidumbre del impacto en esos objetivos y las preferencias particulares del inversionista (incluyendo su actitud ante el riesgo) deben ser consideradas con la finalidad de obtener soluciones satisfactorias. Por último, el conocimiento imperfecto en las preferencias del inversionista debe ser incorporado al proceso de modelado para que el método sea capaz de reproducir efectivamente las decisiones del inversionista. El propósito de esta tesis de doctorado es proponer y validar un enfoque que considera estas situaciones al seleccionar la distribución de recursos maximizando el impacto en los objetivos del inversionista.

Esta tesis modela el conocimiento imperfecto en las preferencias del inversionista, así como la incertidumbre sobre el impacto en los objetivos, sobre la base de la Teoría de Intervalos. Proponemos caracterizar las alternativas de solución al Problema de Optimización de la Cartera usando intervalos probabilísticos de confianza como medida de riesgo y para identificar el comportamiento del inversionista ante el riesgo. Proponemos una obtención indirecta a través de un *Análisis de Desagregación de Preferencias basado en intervalos* con la finalidad de realizar una aproximación a las preferencias reales del inversionista. Usando estas preferencias el método aquí propuesto es entonces capaz de realizar una presión selectiva hacia las soluciones más preferidas por el inversionista en un proceso de optimización. En esta tesis adaptamos la bien conocida optimización evolutiva multiobjetivo con la finalidad de lidiar con parámetros definidos como números intervalo.

El nuevo método ha sido probado extensamente usando las pruebas más comunes de la literatura con datos controlados y datos reales en pruebas dentro y fuera de la muestra. La medida de riesgo y la caracterización de carteras en forma de intervalos de confianza son evaluados en el contexto de la selección de carteras de acciones, comparando el desempeño de nuestro enfoque con varios puntos de referencia en escenarios históricos reales. Los resultados indican que nuestra propuesta supera evidentemente a los puntos de referencia y que es capaz de incorporar satisfactoriamente el comportamiento del inversionista ante el riesgo. Experimentación adicional mostró que la obtención de preferencias permite reproducir las decisiones del inversionista con una gran efectividad, en la mayoría de los casos con más 99% de efectividad promedio para carteras descritas por hasta 12 criterios. Una experimentación final demostró que el uso de la aproximación a las preferencias del inversionista permitió al enfoque propuesto encontrar soluciones que son más preferidas que las soluciones de referencia.

Chapter 1

Introduction

1.1 Context

A problem faced by most organizations and individual investors is how to distribute a monetary amount among a set of investment objects in such a way as to maximize the impact on their objectives. The process of allocating resources maximizing the impact on the investor's (decision maker) objectives is known as Portfolio Optimization, a stage of Portfolio Selection. Portfolio Selection may be divided into two stages [195]. The first stage begins with observation and experience, and ends with beliefs about the future performances of investment objects. While the second stage begins with the beliefs about future performances and ends with the choice of a proportion of resources allocated to each of the investment objects. Here, we focus on the second stage.

Since the 1970s we have seen an accelerated evolution in several fields of science, such as finance, optimization and decision making. It is due to the evolution in these fields that various researchers have made significant advances in Information Theory, generating new products and financial services. However, there is still a collection of challenges that leads to an increasingly large number of papers that consider

- multiple conflicting objectives,
- analysis of the investment objects' performances,
- selection of the *best* investment objects,
- risk management,
- the specific risk behavior of the decision maker, and

- the particular preferences of the decision maker.

All this aspects can be considered in the Portfolio Optimization Problem, which is one of the most addressed problems in Operations Research literature [232,312]. It is a multifaceted problem that poses a number of interesting algorithmic and modeling challenges (e.g., risk consideration, multiple objectives, realistic constraints, preferences modeling), and it is relevant in various contexts including the allocation of resources to financial objects (such as stocks and funds) as well as in the context of non-financial objects (such as projects). Regarding the allocation of resources to financial objects, Portfolio Optimization plays a fundamental role in supporting stocks, a type of security that represents ownership in a corporation. Given that the stock market is one of the most important ways for companies to raise money, it is considered by some authors as "crucial to the existence of capitalism and private property" [241]. The World Bank estimates the value of stocks traded worldwide in 2017 over 77.5 trillion USD[1] [272].

The problem consists in finding (a "good" approximation to) the best assignment of resources to a finite set of investment objects in such a way as to optimize the investor's objectives. Of course, the impact on these objectives depends on the proportion of resources assigned to each investment object and the expected impact of such objects. Note that depending on the context, the Portfolio Optimization Problem is easy to solve. For example if we just want to maximize expected return (cf. Subsection 2.1.1 to see the definition of return), we simply put as much money as possible into the highest returning investment object. The reason that there is some work on this subject is that generally there is a requirement to control risk, which is better achieved when supporting several investment objects (cf. Section 2.1). Therefore, distributing a monetary amount among a set of investment objects in order to maximize the impact on the investor's objectives requires the estimation of the *joint* impact produced by the supported investment objects, as well as the *joint* risk of not achieving such impact. This is due to the concept of risk diversification. It is traditionally based on the idea that the riskiness involved in a given investment depends on the correlation of its constituents, not only on the average riskiness of its separate holdings. Even when diversification is not too relevant in some scenarios (e.g., when the investment objects behave like certain so-called stable Paretian distributions) [93], generally, most practitioners agree that a certain level of diversification is achievable [90]. Thus, the idea of evaluating the distribution of resources in terms of portfolios is opposed to the belief that investors should invest in the (individual) investment objects that offer the highest future impact. Furthermore, we assume here that investment objects are correctly assessed and concentrate on how to select the best portfolio in terms of the investor's preferences.

Finding the best portfolio implies specifying the proportion of resources to support each investment

[1] The value of stocks traded is the total number of stocks traded multiplied by their respective matching prices. Only one side of the transaction is considered. Data are end of year values converted to U.S. dollars using corresponding year-end foreign exchange rates[272].

object in such a way that the best trade-off among the impacts on the decision maker's (DM) objectives is achieved. Thus, the approaches used to find the best portfolio must necessarily consider the DM's perspective. However, defining this trade-off is not trivial since it requires the specification of the DM's preference model and "the elicitation of a preference model's parameters frequently comprises some part of arbitrariness, imprecision, and ill-determination" [97]; thus, the preference modeling has to take into account imperfect knowledge about the DM's decision policy [246]. Several lines of thought dealing with the DM's preference model have been established in the related literature. From these, multicriteria decision aiding (MCDA) provides a wide range of appropriate methods for choosing, ranking and sorting (ordinal classification) problematics. However, the aid provided by MCDA is not effective unless the implemented aggregation model appropriately represents the DM's preferences (implicitly incorporated in a set of decisions made/accepted by the DM). This does not necessarily mean that such preferences must be exactly known; rather, it means that a good approximation to the DM's preferences are required. The effectiveness in such approximation is usually measured on the basis of its ability to reproduce known holistic decisions made by the DM.

Another crucial aspect of the approaches used to find the best portfolio is their ability to approximate to reality. In modern society, many objectives are commonly contemplated when allocating resources [15,270,309,310]. From all these objectives, the most outstanding one is maximizing profit through the portfolio's return [264]; this is sometimes the only objective optimized during the allocation of resources. The optimization of this objective is particularly difficult since estimating future returns (e.g., from time series of historical return data) is very challenging; actually, for some authors it is considered as practically impossible (cf. [48,52,202]). One of the most outstanding arguments in this sense is the one stated by Merton in [202], who indicated that "attempting to estimate the expected return on the market is to embark on a fool's errand". Conversely, many authors have found empirical evidence of a positive risk-return trade-off relation supporting the development of methods to construct portfolios (see, for example, [88,124,126,264,287]). In any case, the high relevance of a correct distribution of resources makes the scientific research about the construction of portfolios a required activity. Throughout this document we will use illustrative examples of the application of our proposals where the only objective to be optimized is maximization of the portfolio's return. However, the reader is advised to consider that such proposals can be applied to other situations; for example, Subsection 2.1.2 presents some alternative objectives commonly optimized when addressing the Portfolio Optimization Problem.

Given the high complexity involved during the estimation of the impact in maximizing the portfolio's return, many underlying criteria are often used to describe this objective. This scenario is similar for all cases where the impact of the portfolios in the objectives can not be exactly known (i.e., for risky objectives). Therefore, approaches capable to deal with many criteria are imperative in finding the best portfolio; even in situations where there is only one risky objective as in the maximization of return. In this particular case, the need for many criteria originates from the DM being unwilling to accept that the uncertainty of future returns can be fully encompassed by a few criteria. Such uncertainty is usually handled by the literature using risk measures from the Probability Theory. However, even in presence of the same level of risk, two decision makers with different attitudes facing risk might have different levels of satisfaction. Hence, besides a risk measure, the approaches finding the best portfolios must incorporate the DM's attitude in presence of risk during the Portfolio Optimization.

From the tools normally used for Portfolio Optimization, the so-called multiobjective evolutionary algorithms (MOEAs) are the most outstanding ones when realistic constraints are imposed (cf. Subsection 1.2). MOEAs are used to solve problems characterized by having multiple conflicting objectives, problems with considerably large search spaces and whose solutions require risk management or uncertainty (Subsection 2.4.2). Problems for selection of alternatives in the presence of constraints deal with combinatorial optimization. Several studies have shown that Genetic Algorithms (a type of evolutionary algorithm) can efficiently find solutions close to the optimum or even optimum for some problems of combinatorial optimization. An important advantage of these algorithms is their ability to obtain an approximation to the Pareto front in a single run [58]. Fernandez et al. [95,96,98] have successfully applied these techniques in the Project Portfolio Optimization context.

1.2 Problem formalization

We define a portfolio x as the vector $x = [x_1, x_2, \cdots, x_n]^\top$, $x_i \in \mathbb{R}$, in the decision space that specifies the proportions of money to invest in n investment objects. The image of a portfolio in the *objective* space is a vector that represents the impact on M objectives established by the decision maker. Whereas the image of a portfolio in the *criteria* space is a vector that represents the impact on k criteria underlying the DM's original objectives. The Portfolio Optimization Problem is to select the feasible portfolio that maximizes the impact on k criteria (representing non-risky objectives, and underlying risky objectives). Formally:

$$\underset{x \in \Omega}{\text{maximize}} \{g(x) = (g_1(x), g_2(x), \cdots, g_k(x))\}. \tag{1.2.1}$$

Where $g_j(x)$ is the impact of portfolio x on the jth criterion and Ω is the set of feasible portfolios (set of portfolios that fulfill the constraints).

Suppose now that there is a maximum number of objects to be supported by portfolio x, N_x, that $y_j = 1$ if $x_j > 0$ and $y_j = 0$ otherwise, and that $l_j \leq u_j$ for some $j = 1, \cdots, n$. Some common constraints applied to Problem (1.2.1) are the following:

$\sum x_j = 1 \longrightarrow$ budget constraint;

$x_j \geq 0 \longrightarrow$ non-negativity/no short-sales constraint;

$l_j y_j \leq x_j \leq u_j y_j \longrightarrow$ individual limits;

$\sum y_j \leq N_x \longrightarrow$ cardinality constraint;

$y_j \in \{0, 1\}$;

$j = 1, \cdots, n$.

A concrete classical formulation of the Portfolio Problem is the one stated by Markowitz in Ref. [195] (Subsection 2.1.1), where the portfolio's expected return is maximized while its variance is minimized. The only constraints considered there are the budget constraint and the non-negativity constraint. Following the results of several research works (e.g., [18,36,259]), it appears that the computational complexity for obtaining the solution of the classical Markowitz model or by several other of its refinements is much lower than the one required by models containing some of the rest of constraints presented above [55]. According to [55], "this practical difference in computational complexity is also theoretically justified by the fact that the classical Markowitz model is a convex quadratic programming problem that has a polynomial worst-case complexity bound, while the other formulations are usually modeled by adding binary variables, thus becoming mixed integer quadratic programming problems, considerably more difficult that can not be solved by exhaustive methods".

1.3 Background

In 1952, the now-called Modern Portfolio Theory was founded by Markowitz in [195]. The only objective being optimized in that work is the maximization of the portfolio's return. The main contribution of Markowitz's work was the formalization of the Portfolio Optimization Problem with the argument that for any two portfolios with the same expected return, the decision maker prefers the portfolio with the lowest risk[2]. This proposal has been the main idea in most theoretical research on the Portfolio Optimization Problem, and it is still active since it is sometimes used as a classical benchmark. However, its application in practice has been rather scarce due mainly to some relevant limitations (Subsection 2.1.2).

Several authors have proposed different alternatives to overcome the limitations of the mean-variance model. For example, Markowitz proposed in [196] to substitute the variance for the semi-variance. In this way, some drawbacks related to the variance as the risk measure are discarded from the model. Nonetheless, other limitations continue present, such as a poor modeling of the DM's attitude facing risk. Robust optimization [32,266] has been applied by some authors (e.g., [115,162,233]) with the intention of solving

[2]Although some authors refer to uncertainty and risk as synonyms, in this work we will treat the concept of risk as the uncertainty that negatively affects the decision maker.

some problems caused by uncertainty of the impacts on the objectives. Robust optimization implicitly considers that these impacts have been estimated with errors and uses an interesting concept called *uncertainty set* to protect the results of the approach against the worst scenarios. So, the resultant portfolio tends to be more stable and less sensitive to changes of the approach's estimations (cf. [88]). Nevertheless, such protection could be considered by the DM as too pessimistic and can result in under performance of the portfolios when the estimated values tend to the "true" parameter values and/or in situations where the investment objects returns in the portfolio tend to grow. With respect to the risk of not achieving the portfolios' expected returns, some authors have proposed to incorporate higher statistical moments, such as skewness and kurtosis, in order to better describe the probability distribution of the portfolio's return (e.g., [76,130,252,256]). However, the incorporation to the model of the DM's risk attitude using such statistical specific-knowledge tools is too complicated. To avoid this limitation, some authors have used probabilistic quantiles to provide valuable information to the DM (see e.g., [20,126,152]). From these, Greco et al. propose in [126] to explicitly use quantiles as criteria to maximize. Such approach can deal with virtually any probability distribution, consider higher statistical moments and use many quantiles in order to better describe the probability distribution of the portfolio's return. However, in the pursuit of a better description of the probability distribution, the quantity of criteria could be so high that it exceeds the cognitive capacity of the DM (cf. [207]).

On the other hand, information about the DM's preferences can be obtained either directly or indirectly. In a direct elicitation method, the DM, generally in collaboration with a decision analyst, is in charge of making a direct setting of the parameter values of the preference model. The direct elicitation approach has been considered as less adequate for elicitation or assessment purposes (see, for example, [47]). Some arguments against direct elicitation are the following: i) the preference model's parameters are meaningless as long as the multicriteria aggregation procedure in which they are used has not been specified; ii) the holistic information provided by the DM's judgments when she/he compares pairs of actions, or assigns actions to classes, is more suitable and valid; iii) the DM may not be accessible (e.g., the CEO of a multinational company) or may be an ill-defined entity (e.g. a heterogeneous group); iv) the DM usually has difficulties to explicitly specify numerical parameters and the time and cognitive effort required to do so may be inhibitory. Indirect elicitation approaches have been used for decades to build functional or utility decision models (e.g., [156,226,268]). Nevertheless, using indirect elicitation methods can not avoid certain imperfect information in setting the model's parameters. According to Roy et al., in Ref. [246], this *imperfect knowledge* must be taken into consideration by any approach modeling the DM's preferences.

Finally, since the 1990s, an increasing number of works have proposed the application of MOEAs to address the Portfolio Optimization Problem. For example, Lin and Gen [182] use the Markowitz model as a basic mathematical model, maximizing return and minimizing risk. In their work, they argue to have proven the reliability and efficiency of genetic algorithms in the optimization of stock portfolios. Shoaf and Foster [261] applied genetic algorithms to Markowitz's portfolio selection problem and found that, under certain assumptions, the temporal complexity of the genetic algorithms approximates to $O(n \cdot log n)$. The results obtained are interesting and confirm the efficiency of the genetic algorithms due to their rapid convergence towards the best solutions since the first runs of the algorithm and their satisfactory calculation time. Saborido et al. [250] compare the results of three genetic algorithms in addressing a constrained three-objective optimization problem in order to analyze the efficient portfolios which optimize the three criteria simultaneously. From the genetic algorithms used in the comparison, the so-called multiobjective evolutionary algorithm based on decomposition (MOEA/D) has been the basic technique for several works (e.g., [296]) and a common benchmark to contrast results (e.g., [177,210,250]). The version of this algorithm using Differential Evolution, MOEA/D-DE, has also been applied in the context of the Portfolio Optimization Problem (cf. [54]).

1.4 Research questions

Given the discussion presented in the previous sections, we enumerate the following research questions as, from our perspective, the most relevant unanswered questions in the related literature.

1. How to manage risk in the objectives whose impacts generated by the portfolios cannot be exactly known, in such a way as to incorporate the decision maker's attitude during Portfolio Optimization?
2. How to model the decision maker's preferences in such a way that the proposed approach can reproduce his/her holistic decisions?
3. How to select the portfolios that are more preferred by the decision maker when many criteria are considered?

1.5 Overall objective

To propose and validate an approach that addresses the Portfolio Optimization Problem finding portfolios that are most preferred by the decision maker.

1.6 Specific objectives

i. To propose an appropriate way to model the DM's attitude facing risk.

ii. To create an elicitation model that builds the decision maker's preferences considering his/her imperfect knowledge.

iii. To consider the imperfect knowledge in the preferences of the decision maker during the search for the best portfolio.

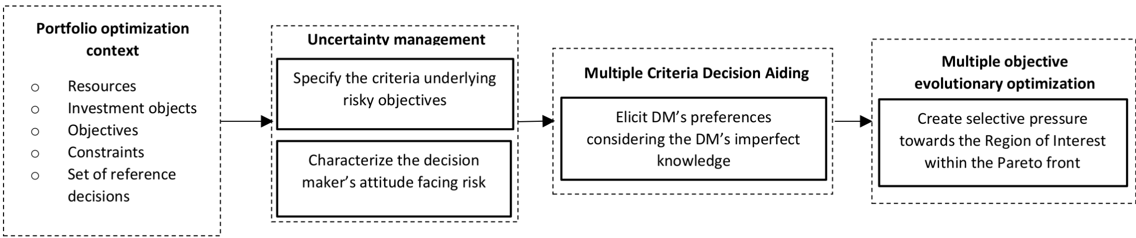
iv. To ensure that the approach has the capacity to handle many criteria.

v. To improve the selective pressure towards the DM's most preferred portfolios.

1.7 Methodological proposal

We show here the sequence of steps of our approach, which depends on three basic pillars: Uncertainty Management, Multicriteria Decision Aiding and Multiobjective Evolutionary Optimization. Figure 1.1 shows this sequence of steps.

Figure 1.1: Proposed methodology



Portfolio Optimization context. Here, some of the inputs required by the proposed approach must be provided. These inputs are the *resources* that will be allocated to the (finite) set of *assets* or *investment objects* to be supported, the *objectives* that the decision maker wants to optimize and the *constraints* that must be fulfilled during the optimization. The approach also requires a set of reference holistic decisions made/accepted by the decision maker in order for the approach to obtain an approximation to his/her system of preferences.

Uncertainty management. Our proposal uses probabilistic confidence intervals as underlying criteria to the objectives whose impacts are not exactly known (risky objectives). Such characterization allows the investor to consider not only the expected impact of the portfolios but also the risk of not obtaining that expected impact. This approach identifies the behavior of the investor when facing risk, and provides her/him aid depending on her/his own behavior facing risk.

Multicriteria decision aiding. Here, we use an indirect elicitation method to infer the decision model's parameter values from holistic decisions provided by the DM. Our proposal is to implement regression-like methods to create a decision model as consistent as possible with the set of reference (training) decisions made/accepted by the DM so her/his decisions can be reproduced.

Multiobjective evolutionary optimization. Portfolio Optimization is performed on the basis of one of the most outstanding multiobjective optimization approaches, the so-called multiobjective evolutionary algorithms (MOEAs). To the best of our knowledge, MOEAs have not been used in the context of Interval Theory except in Refs. [23,264]. In this work, some MOEAs have been enhanced so that they can deal with alleles and fitness values described as interval numbers.

1.8 Hypothesis

This work is based on the following hypothesis:

H.1 The proposed approach builds portfolios that are more preferred by the decision maker than the reference portfolios.

1.9 Document structure and expected contribution

We close this chapter by outlining the thesis organization. In doing this, we also underline the proposed research's novelties and contributions to advance in answering the questions raised above.

Chapter 2. The fundamental background material and notation are presented in Chapter 2, which is divided into five parts. In the first part, Section 2.1, we present what we consider the most relevant aspects of Portfolio Optimization. We start there by providing the classical formulations of Modern Portfolio Theory (MPT). Later, we mention some alternative approaches and objectives defined beyond the MPT. After that, we describe and discuss some common uncertainty management techniques. Finally, we provide a typical application of the Portfolio Optimization Problem, the optimization of stock portfolios, and some ways to evaluate individual stocks and stock portfolios. In the second part, Section 2.2, the basic ideas of the so-called Interval Theory are presented; this theory contains one of the main concepts used in this thesis, the interval number. In Section 2.3, the multicriteria problems and processes are revised. There, we start by providing some basic definitions. Later, we revise the main schools of decisions, namely, the normative and relational approaches. This section provides a detailed description of an approach capable of modeling the investor's preferences considering uncertainty, vagueness, and/or ill-determination, the so-called interval-based outranking approach. Finally, we address the issue of how to elicit the DM's preferences, and focus on the so-called Preference Disaggregation Analysis. In Section 2.4 we describe in a general way the type of meta-heuristics most used in the literature related to Portfolio Optimization, the Evolutionary Algorithms. We detail there the characteristics of what is considered the most outstanding multiobjective Evolutionary Algorithm based on decomposition, MOEA/D, and revise the literature related to preference-based Evolutionary Algorithms. In the last part of this chapter, Section 2.5, Compensatory Fuzzy Logic is presented as a logical model that allows modeling decision-making processes. That section focuses on Compensatory Fuzzy Logic based on the geometric mean, which presents several interesting properties.

Chapter 3. Chapter 3 presents our contributions on managing uncertainty on the impacts of risky objectives. Particularly, we follow the tendency in the literature and assume that an approximation to the probability distribution of the impacts can be obtained. We propose to (partially) characterize portfolios through confidence intervals using them as underlying criteria to the risky objectives. This characterization allows not only to consider uncertainty of the impact on risky objectives but the decision maker's attitude in presence of risk. An extensive evaluation of this way to characterize portfolios and its capacity to incorporate the decision maker's attitude in presence of risk is performed in the context of stock Portfolio Optimization.

Chapter 4. We present our proposal to elicit the decision maker's preferences in Chapter 4. There, an indirect elicitation procedure based on the so-called Preference Disaggregation Analysis is used. We consider the imperfect knowledge of the decision maker and assume that arbitrariness, imprecision, ill-determination and uncertainty are involved in the elicitation procedure. Thus, Interval Theory is used to model the values of the decision parameters representing the DM's preference model. We indirectly elicit these values from holistic decisions made/accepted by the decision maker. Moreover, with the goal of further simplification of the DM's work, the only information that she/he has to provide is a set of portfolios assigned to some preferentially ordered categories. From this assignments, the proposed approach is able to infer binary preference relations between pairs of portfolios; and, by minimizing the inconsistencies with these inferences, eventually it finds a preference model through which it is possible to effectively reproduce the DM's decisions. Such effectiveness is analyzed in-sample and out-of-sample with portfolios characterized by up to twelve criteria.

Chapter 5. In Chapter 5, we use Fuzzy Logic and the approximation to the DM's preferences to aggregate the impacts on the criteria (underlying both risky and non-risky objectives). The approach uses the decision maker's preferences to create a selective pressure towards the most preferred portfolios. This way, the complexity of the DM's final decision is reduced (above all) when there are many criteria in the optimization problem. We assess the proposed approach's performance by addressing the stock Portfolio Optimization Problem, where a novel way to use the so-called fundamental and technical analysis is presented to characterize the portfolios.

Chapter 6 The thesis concludes in Chapter 6, summarizing our contributions and presenting future research directions.

Chapter 2

Theoretical framework

2.1 Portfolio Optimization

Even though maximization of return is the only objective optimized in the conventional resources allocation (e.g., [195]), the formulations with multiple criteria are also often mentioned in the literature (see e.g., [234,269]). There are two main reasons for this situation [311]. The first is that the decision maker (DM) makes considerations additionally to return, such as social responsibility, liquidity and the proportion of resources allocated to certain kinds of objects. That is, instead of being interested only in maximizing the portfolio's return, the decision maker may be interested in optimizing several objectives at the same time. The second reason for using a multicriteria formulation is that, even when contemplating just the maximization of return, the DM is not willing to accept the assumption that the uncertainty of the actual return can be fully encompassed in a single criterion. Not even through a "reliable" estimation of the return as it is the expected value. Consequently, the decision maker wants the selection of the best solution alternative to be made on the basis of additional estimations such as financial indicators and expert opinions.

In this section, we first provide an outline of what is considered a pathbreaking in the Portfolio Optimization Problem maximizing the portfolio's return, Modern Portfolio Theory (MPT), together with some of its most important limitations. Later, we go beyond MPT and mention some of the most common objectives alternative to maximization of return. After that, some interesting properties and characteristics of risk measures are discussed. Finally, a common application of the Portfolio Optimization Problem to stock portfolios is described.

2.1.1 Modern Portfolio Theory

In 1952, Markowitz's work laid the foundation in Ref. [195] of what is now known as Modern Portfolio Theory. The main relevance of Markowitz's work is the argument that for two portfolios with the same level of expected return, a rational decision maker must choose the portfolio with the lowest variance. Several formulations of Markowitz's proposal can be found in the literature. Let us now provide these formulations.

Classical formulations

Let $\Omega \subseteq \mathbb{R}^n$ be the set of feasible portfolios, then $x \in \Omega$ means that portfolio x fulfills the constraints imposed in the model. The problem then leads to the search for the best assignment of values to the components of $x = [x_1, \cdots, x_n]^\top$, where x_j (generally in [0,1]) is the proportion of resources to be allocated to the jth investment object. These objects have returns $r_1^{t+1}, \cdots, r_n^{t+1}$, denoted by the vector $r = [r_1^{t+1}, \cdots, r_n^{t+1}]^\top$. The return of each object is the percentage change in its value over a given time period; that is

$$r_j^{t+1} = \frac{p_{t+1}^j - p_t^j}{p_t^j},$$

where r_j^{t+1} is the return of the jth object in the portfolio, and p_t^j is the price of the object in time period t.

Of course, negative returns are possible. In the case of a stock, its market price can vary both up and down due to company performance and general market conditions. If one knows the exact value of each r_j^{t+1}, one can easily compute the return of portfolio x, $R(x)$, as (cf. [90]):

$$R(x) = \sum_{j=1}^{n} x_j r_j^{t+1}.$$

Given that the portfolio return depends on future events, it will normally be uncertain. This is the reason why the *expected* return is important rather than the *actual* return. It is therefore assumed that $R(x)$ follows a density function f. And the expected return of portfolio x is

$$\mathbb{E}(R(x)) = \int R(x) f(R(x)) dR(x).$$

Moreover, for a given return value β, a feasible portfolio is at the efficient frontier if it is a solution to the following problem:

$$\underset{x\in\Omega}{\text{minimize}}\ \sigma^2(R(x)) \tag{2.1.1}$$

subject to the constraints

$$\mathbb{E}(R(x)) = \beta,$$

$$\sum_{j=1}^{n} x_j = 1.$$

Where $\sigma^2(R(x))$ is the return's variance of portfolio x and $\mathbb{E}(R(x))$ is its expected value. We refer to this version of the problem as the formulation of risk minimization.

Equivalently, for a given variance value α, a feasible portfolio x is at the efficient frontier if it is a solution to the following problem:

$$\underset{x \in \Omega}{\text{maximize}}\ \mathbb{E}(R(x)) \tag{2.1.2}$$

subject to the constraints

$$\sigma^2(R(x)) = \alpha,$$

$$\sum_{j=1}^{n} x_j = 1.$$

We refer to this version of the problem as the expected return maximization formulation.

We denote by σ_i the standard deviation of r_i^{t+1}, by ρ_{ij} the correlation coefficient of the returns of the objects x_i and x_j (for $i \neq j$), and by Σ the matrix $n \times n$ of co-variances of the returns of all the variables, i. e.,

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix}.$$

Where $\sigma_{ii} = \sigma_i^2$ and $\sigma_{ij} = \sigma_{ji} = \rho_{ij}\sigma_i\sigma_j$ (for $i \neq j$). Normally, it is assumed that Σ is positively defined, i.e.,

$$[x_1, \cdots, x_n]^\top \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} > 0,$$

for all $x \neq 0$.

So, for a given portfolio x, it is possible to calculate its variance as

$$\sigma^2(R(x)) = x^\top \Sigma x.$$

It is possible to obtain an alternative formulation of the portfolio problem called risk aversion formulation as:

$$\underset{x \in \Omega}{\text{maximize}}\ x^\top \zeta - \psi x^\top \Sigma x \tag{2.1.3}$$

subject to

$$x^\top \iota = 1; \quad \iota^\top = [1, 1, \cdots, 1].$$

Where $\zeta = \{\mathbb{E}(r_1), \cdots, \mathbb{E}(r_n)\}^\top$ and ψ is an indicator of risk aversion that penalizes the performance of the solution according to its risk; i.e., ψ is a parameter that represents the specific aversion to risk of the decision maker, and determines the compensation between the expected return and the risk of the portfolio.

2.1.2 Beyond Modern Portfolio Theory

Here, we mention some limitations of Modern Portfolio Theory, these are considered as some of the most relevant limitations in the related literature. We also briefly describe some alternative approaches capable to outperform some of these limitations.

Limitations of Modern Portfolio Theory

The approach proposed by Markowitz opened new directions in the way that decisions to solve the Portfolio Optimization Problem are taken and modeled. However, today more than 60 years after its publication, this approach is used considerably more in theory than in practice [162]. This is due to several disadvantages of the model that arise when it is used in practice. Some of the most important limitations of the mean-variance model are:

- The model assumes normality in the distribution of the portfolios' returns.
- Variance as a risk measure has some undesirable characteristics.
- The model makes a poor modeling of the decision maker's attitude facing risk.

The approach proposed by Markowitz assumes that either returns are jointly normally distributed, or that all DMs only care about the mean and the variance of their portfolios [90]. However, empirical evidence suggests that distributions of returns typically have heavier tails than those that are implied by the normal distribution, and are often not symmetric with respect to the mean (cf. [163,208,225,276]).

An alternative is to extend the traditional mean-variance framework by directly incorporating new objectives that consider, for example, skewness and kurtosis (see [252]). The idea is that a rational DM, if requested to choose between two alternatives with the same mean and variance, will probably choose the alternative with the highest skewness and the lowest level of kurtosis (cf. [76,256]). Harvey and Siddique in Ref. [130] show that skewness may be relevant in the stock Portfolio Optimization Problem. They establish that if the returns of the stocks exhibit non-diversifiable co-skewness, then the DM would expect a higher level of return. On the contrary, if there is a positive skewness in the distribution of these returns, the DM would be willing to accept low expected returns.

The works of Levy and Markowitz [174] and Kroll et al. [167] reveals that the selection of objects in a stationary environment using the mean-variance approach results in portfolios that are very similar to those formed by a direct optimization of the expected utility, and thus they suggest that the highest moments do not play an important role in practice. However, Cremers et al. [64,65] demonstrate empirically that there are some probability distributions that are insensitive to the highest moments and, therefore, the mean-variance approach behaves well when the considered returns follow these distributions. However, this result does not hold true when the returns do not behave in such a way, making the approach proposed by Markowitz to perform poorly in these cases.

On the other hand, regarding the uncertainty of not attaining the portfolio's expected return, some authors have proposed diverse definitions of risk because different DMs adopt different investment strategies to reach their investment objectives. In a sense, risk is itself a subjective and relative concept [24,33], which is probably the main characteristic of risk. In turn, this leads us to believe that, even if we can identify some desirable characteristics that our risk measure should have, this does not ensure that this risk measure can guarantee solving the problem of representing all DMs' attitude when facing risk.

The first risk measure proposed in portfolio theory to control the risk of a portfolio is variance. Variance was chosen due to its easy calculation and that, together with the mean of the distribution, it contains all the relevant information about the returns if they are normally distributed.

Therefore, if we do not make the assumption that returns follow a normal distribution, then using the variance as a risk measure does not guarantee that we can control the risk of a portfolio. Additionally, using the variance for this purpose presents several problems, the first of which is that it contemplates the negative deviation (from the mean) that is not desirable, but also considers the positive deviation, which, in turn, is desirable. This generates some adverse effects in the mean-variance model. For example, the model fails to meet some monotonicity properties, such as that the DM using this model can choose an object that provides less *satisfaction* than another object. This can happen when an additional unit of return increases the expected return of a portfolio, causing the dispersion of returns to be greater, which increases the variance of the portfolio. For example, suppose portfolios a and b with the following returns.

r_1	r_2	r_3	r_4	r_5
25	25	25	25	25

Table 2.1: Returns of portfolio a

Although the increase in the expected return of portfolio b is desirable, the mean-variance model selects portfolio a as the best alternative, since the positive influence of the return of the portfolio b is compensated by its variance.

Fabozzi et al. in Ref. [89] show that decision makers use a variety of risk measures and approaches

r_1	r_2	r_3	r_4	r_6
25	25	25	25	26

Table 2.2: Returns of portfolio b

beyond the traditional framework of Markowitz's mean-variance. The concept and measurement of risk have been active areas of research and debate in the last two decades. Some researchers have proposed new risk measures that consider only negative deviations, among which the so-called value at risk [152] stands out. This method can be considered as the worst $\alpha\%$ quantile of returns. Although using this new risk measure is popular, it has several undesirable mathematical characteristics such as non-subadivity and non-convexity [240]. This led to the introduction of the so-called coherent risk measures [20], among which the conditional risk value [240], that can be defined as the average of the worst $\alpha\%$ of the cases, stands out. These and other risk measures are presented and discussed in Subsection 2.1.3.

Alternative objectives

Jensen [147] argues that the interests of the investor regarding finances should be aggregated into a single objective, maximization of the market value. The justification for such an argument comes from the "impossibility of maximizing in more dimensions and that doing so leaves the investor without an objective". The alternative perspective establishes that investors objectives (e.g., in the case of organizations) must represent the interests of all stakeholders (e.g., employees, customers, suppliers, shareholders, the community, etc.), instead of focusing only on the shareholders of the organization [108]. An illustrative example is proposed by Zopounidis et al. in Ref. [312]. Consider an organization such as a bank, whose operation depends on a wide range of very different and complex processes, including credit management (global risk management policies, credit rating, loan setting, etc.), the organization of branches, internal audit and control and relationships management with clients. Inevitably, operational decisions for all these activities are based on multiple decision criteria (and constraints), even if the overall objective of the organization is to maximize the wealth of its shareholders.

According to Zopounidis et al. [312], the finances of the organization also acquire a multicriteria value. For example, Graham and Harvey in Ref. [125] surveyed 392 financial managers of US firms (United States and Canada) and found that executives rely on practical and informal rules when choosing the capital structure, focusing mainly on issues such as liquidity of the objects that indicate financial flexibility in the portfolio, earnings per object and some market indicators such as the recent appreciation of shares prices.

The liquidity of a portfolio consists of the ease/speed with which its objects can be sold. The most important advantages of liquidity are: i) it allows the DM to take advantage of temporary investment opportunities, ii) it offers the opportunity to leave harmful investments, and iii) it provides the ability to respond promptly to changes in the DM's behavior facing risk. Moreover, liquidity influences the preferences of the DM when forming the portfolios. For example, although some foreign objects may represent lower investment opportunities, they sometimes tend to have larger trading volumes than domestic objects [114]. This is because the former have desirable liquidity properties that domestic objects do not have. In fact, Ang et al. [13] showed that non-liquidity increases risk aversion and distorts object allocation. The most reliable measure of liquidity is the volume of objects trading, which is considered in virtually all financial markets.

On the other hand, since the 2008 crisis, investors tend to select allocation strategies that are increasingly oriented towards social responsibility [279]. The amounts already invested in socially responsible funds point to the demand for these products. Therefore, it is clear that there are investors with other preferences, in addition to the classical financial ones. Furthermore, with the existence of agencies that evaluate the social responsibility efforts of the companies, it is possible to carry out studies to investigate whether the portfolios actually incorporate social responsibility in their operations. According to the Social Investment Forum [237], "at the start of 2012, in the US, professionally managed assets following socially responsible investment strategies stood at $3.74 trillion, a rise of more than 486 percent from $639 billion in 1995. Over the same period, the broader universe of assets under professional management increased only 260 percent from $7 trillion to $25.2 trillion". One way to consider social responsibility and ethics in the selection of the best portfolio is on the basis of expert's analyses.

Finally, the need to modify the portfolio structure arises from the changing environment in which the decision-making process is usually found in the context of the Portfolio Optimization Problem. However, the frequent change of the investment means a detriment to the real financial gain of the portfolio. In other words, the optimization strategies are sensitive to the transaction costs required to form the portfolios [223]. Costs in transactions can be seen as the expenses in which the DM incurs with the purpose of changing the portfolio structure. Thus, the ideal situation to maximize the final return of the portfolio is to find a balance between the expected return and the cost of transactions. The most common methodology to consider this balance is to use the difference between the current portfolio and the "optimized" portfolio.

All these lines of thought show a growing tendency to consider more and more objectives in the portfolio selection process (see [270], [310], [309], [14], [15]). Some examples of the most mentioned objectives in literature are

- maximization of return [197], [250],
- maximization of social responsibility and ethical considerations [123], [279], [128],
- maximization of liquidity[9], [158], [299],
- maximization of dividends [135],

- maximization of the return with respect to a reference point [270],
- maximization of the return in different periods of time (short, medium, long term),
- maximization of the amount invested in Research and Development (R&D) [270], and
- minimization of transaction costs[193], [297].

2.1.3 Uncertainty management

During the process of portfolio optimization, the estimation of future events (e.g., forecasting the investment objects returns) necessarily implies uncertain consequences to the actions performed by the investor. The consequence of any action is determined not just by the action itself but also by a number of external factors. These external factors are both beyond the control of the investor and unknown to her/him at the time of the decision.

Generally, the concepts of uncertainty and risk are considered differently. In the sense of the consequences of the investor's actions, uncertainty means that the *true state* (complete description of the external factors determining the consequences of the possible decisions) is not known before the investor has to make the decision. While risk is something the investor bears and is the outcome of uncertainty; it is how much the investor would be damaged if she/he engages in an uncertain action. Thus, there might be instances where risk and uncertainty are used interchangeably. For instance, suppose a coin-toss game is played where one has to bet \$0.50 and if heads come up one wins \$1, but loses everything if tails appear. The risk here is that one loses everything because the risk is that tails may appear. The uncertainty here is that tails may appear. Given that tails appear, one loses everything; hence, uncertainty brings with it risk. Uncertainty is the possibility of an event occurring, and risk is the ramification of such an event occurring [217].

In the context of the maximization of the portfolio return, we will define a risky portfolio as portfolio with more than one financial consequence, say $c_1, c_2, \cdots, c_n$ and, for at least one value c_i, $0 < \mathbb{P}(c_i) < 1$, where $\mathbb{P}(c_i)$ denotes the probability of c_i occurring. Note that if there is one value such that $0 < \mathbb{P}(c_i) < 1$, there must be at least one more observation, c_j, with $0 < \mathbb{P}(c_j) < 1$. The total probability must be equal to 1, $\sum \mathbb{P}(c_i) = 1$. By this definition, the future value of a risky portfolio may have more than one value c_i (actually at least two values). In practice, the probabilities are unknown but can be easily estimated.

The DM's level of risk aversion when selecting portfolios can be characterized by defining an indifference curve. When the objectives are maximization of return and minimization of risk, then this curve is composed of risk/return pairs that define the trade-off between expected return and risk. That is, it establishes the increase in expected return that must exist so that the DM can consider that the increase in risk is worth it, or vice versa: the risk reduction that must exist to consider that a reduction in the expected return is worth it. This curve defines a function on the DM's behavior when forced to make compensations between risk and expected returns.

The behavior of the DM facing risk can be divided into *risk-averse*, *risk-neutral* and *risk-taker*. Without loss of generality, we say that a DM is risk-averse when he/she is reluctant to support a portfolio that has uncertain expected return instead of supporting another with less risk but with possible expected minor consequences (considering a maximization of the objectives). In terms of expected utility, a DM is risk-averse if and only if his/her utility function is concave. A DM presents a risk-taker behavior when his/her preferences tend to select solutions with better expected return rather than the certainty of those consequences; that is, a DM is risk-taker if he/she prefers to support a portfolio that has certain expected return instead of supporting another portfolio with expected minor consequences although with greater certainty. This implies convexity in the utility function of a risk-taker DM. Finally, it is also possible to define a linear function that represents a neutral behavior of the DM facing risk. We say that a DM presents this type of behavior when his/her attitude is not risk-averse nor risk-taker (i.e., the DM is indifferent between choosing portfolios with the same expected consequences even if one of them represents more risk).

There is a widely accepted convention that, generally, increasing a portfolio return necessarily exposes it to more risk. However, how to quantify this risk is not straightforward. This has implied that there is no risk measure whose use is generalized among investors. Rather, it is common to see new definitions of risk measures trying to overcome (some of) the limitations of previous measures. We turn next to several alternative suggestions which have appeared in the financial and economic literature on how to measure risk related to portfolios.

Risk measures

Let Δ be the set of possible probability distributions of changes in value (risks). A risk measure, ρ assigns to a risk a real number that represents the degree of risk involved i.e., $\rho : \Delta \longrightarrow \mathbb{R}$. We denote the distributions of changes associated to portfolio x by $\mathcal{R}_x$.

It is possible to classify risk measures as belonging to one or more of the following categories:

- Consistent risk measures with respect to stochastic dominance.
- Coherent risk measures.
- Practical risk measures (with respect to the use of resources; for example, the time invested in implementing and applying them).

Consistent risk measures with respect to stochastic dominance Consistency of risk measures is based on the concept of stochastic dominance, which is related to the concept of maximization of expected non-decreasing or concave utility functions of the type Neumann-Morgenstern [281]. In particular, first-order stochastic dominance of a portfolio x over a portfolio y, whose returns are the random variables X and Y and follow distribution functions F_X and F_Y respectively, exists if for any consequence $\alpha \in S \subseteq \mathbb{R}$, x generates at least α with a probability equal to or greater than the probability with which y does, and for some α, x generates with a probability higher than y a value equal to or greater than α; i.e., $\forall\alpha$: $\mathbb{P}(X \geq \alpha) \geq \mathbb{P}(Y \geq \alpha)$ and $\exists\beta$: $\mathbb{P}(X \geq \beta) > \mathbb{P}(Y \geq \beta)$. This can be denoted in terms of the cumulative distribution functions of the two portfolios as $F_X(\alpha) \leq F_Y(\alpha)$, for all α, with strict inequality at some α. This means that the probability of X falling below a specified level α is smaller than that of Y falling below the same level. This is equivalent to saying that any DM maximizing expected utility that prefers more to less always considers x at least as good as y.

First-order stochastic dominance makes assumptions about the cumulative distribution functions and requires the DM to prefer more to less; furthermore, it does not allow one to distinguish between two portfolios with the same mean return. On the other hand, second-order stochastic dominance makes weaker assumptions about the integral of the cumulative distribution functions but requires the investor to be risk-averse as well as prefer more to less. Second-order stochastic dominance may occur when the two portfolios share the same mean return. Portfolio x has second-order stochastic dominance over portfolio y if X implies a lower risk (it is more predictable) and has an average equal to or greater than Y. That is, we shall say that x has second-order stochastic dominance over portfolio y if:

$$\int_{-\infty}^{\alpha} \mathbb{P}(X \leq s)ds \geq \int_{-\infty}^{\alpha} \mathbb{P}(Y \leq s)ds \text{ for all } \alpha \text{ and,}$$

$$\int_{-\infty}^{\beta} \mathbb{P}(X \leq s)ds > \int_{-\infty}^{\beta} \mathbb{P}(Y \leq s)ds \text{ for some } \beta.$$

Second-order stochastic dominance is a weaker condition than first-order stochastic dominance and three situations may rise when comparing two portfolios using these orders of stochastic dominance [153]:

- Nothing can be said.
- Y is first- and second-order stochastic dominated by X. (Or X to Y.)
- Y is second-order stochastic dominated by X but not first-order stochastic dominated. (Or X to Y.)

If a risk measure provides the same order as stochastic dominance for all non-decreasing utility functions, it is said that the risk measure is consistent with first-order stochastic dominance. Similarly, if a risk measure provides the same order as stochastic dominance for all concave utility functions, we call the risk measure consistent with respect to second-order stochastic dominance. The above is assessed for arbitrary functions. Since stochastic dominance is only a partial order then it is possible to say that consistency is a minimum desirable requirement that any risk measure should satisfy [68]. The Portfolio Optimization Problem often involves only certain characteristics of the portfolios' return distributions, such as expected return and risk. In this situation, it is fundamental for the problem to be consistent with the corresponding relationship of stochastic dominance in order to guarantee that its solution is a stochastically non-dominated portfolio. The verification of this consistency is reduced to the choice of a risk measure that is compatible with the respective stochastic dominance relation.

Coherent risk measures After the work of Artzner et al. [20], the measurement of risk took a relevant turn with respect to traditional techniques. They proposed a collection of axioms for risk measurement in such a way that the techniques that fulfilled these axioms were called *coherent risk measures.* Let $\mathcal{R}_x$ and $\mathcal{R}_y$ take values from Δ, a and h take real values, and $h > 0$, then ρ is a consistent risk measure if it is

- Monotone:
$$\mathcal{R}_x \leq \mathcal{R}_y \Longrightarrow \rho(\mathcal{R}_x) \leq \rho(\mathcal{R}_y)$$
- Sub-additive:
$$\rho(\mathcal{R}_x + \mathcal{R}_y) \leq \rho(\mathcal{R}_x) + \rho(\mathcal{R}_y)$$
- Positively homogeneous
$$\rho(h\mathcal{R}_x) = h\rho(\mathcal{R}_x), \text{ and}$$
- Translation invariant
$$\rho(X + a) = \rho(X) - a.$$

Monotonicity is a desirable characteristic of ρ since, for example, if two portfolios x and y have the same initial value yet x always returns more than y (i.e., $\mathcal{R}_x \leq \mathcal{R}_y$), then ρ should reflect this [153].

The sub-additivity property reflects the characteristic that a portfolio built from "sub-portfolios" will have an amount of risk that is at most the sum of the risks of the separate sub-portfolios. The inequality of this property becomes equal when the risk of the portfolio depends completely on the sum of the risks of the sub-portfolios; that is, when the sources of the risks of the sub-portfolios are generated by concurrent events. In the case of a sub-additive measure, diversification of the portfolio always leads to risk reduction; while for measures that do not fulfill this property, diversification can produce an increase in risk even when the partial risks are generated by mutually exclusive events [4].

The property of positive homogeneity indicates that if we multiply everything in our portfolio by the same amount (and hence the associated distribution of value changes) then the risk should grow by the same factor [153]. Positive homogeneity is similar to translation invariability in the sense that the latter considers the case where we add or substract a fixed proportion from the changes of a portfolio. In such a case, ρ should change by the same amount.

Below we mention and describe some of the most used risk measures when addressing the Portfolio Optimization Problem. Of course, the list of techniques described is not exhaustive but it does contain some of the most common risk measures in the related literature.

- Semi-variance (S)
- Value at risk (VaR)
- Conditional risk value ($CVaR$) / Average risk value ($AVaR$) / Expected shortfall (ES)
- Spectral risk measures ($SRMs$)
- Quantiles of the probability distribution

Semi-variance Markowitz in Ref. [196] argues that the analysis based on the semi-variance (S) tends to produce better portfolios than those based on the variance (σ^2) because the DM usually is more concerned about the performance below the expected return than above it, and came to declare the semi-variance as "the best measure to quantify the risk".

Semi-variance is similar to variance, with the difference that the first one considers only the observations below the mean. In other words, semi-variance looks only for the negative fluctuations of a portfolio. The semi-variance is the average of the square of deviations of the values lower than the mean. Formally, if $\overline{X}$ is the mean of n returns r_i then the semi-variance S is defined as:

$$S = \frac{1}{n} \sum_{r_i < \overline{X}}^{n} (\overline{X} - r_i)^2$$

This risk measure has been used in the Portfolio Optimization Problem instead of the variance by Yan and Li [290] and Yan et al. [291], where the selection of portfolios in a single period is extended to multiple periods. It is also used by Najafi and Mushakhian [218], Zhang et al. [297] and Giilpinar et al. [119], incorporating transaction costs. It is possible to show that semi-variance is not a coherent risk measure

Value at risk The objective of Value at Risk (VaR_α) [275] is to find an answer to the question what is the expected loss with an accumulated probability α in a time horizon τ? For a cumulative probability distribution function $F(\cdot)$, a random variable $\omega(\tau)$ representing the return of a portfolio over a period of time τ and an accumulated probability α, then

$$F(\omega(\tau) \leq -VaR_\alpha) = \alpha,$$

hence VaR_α is defined as

$$VaR_\alpha(\omega(\tau)) = -\inf_x \{x|\mathbb{P}(\omega(\tau) \leq x) \geq \alpha)\}$$

For example, suppose a VaR_α = \$1,000.00, with $\alpha = 0.9$ and $\tau = 1$ month. This implies that it is expected with a 90% accumulated probability that the portfolio will suffer a maximum loss of \$1,000 in a period of one month; or, equivalently, there is a 10% cumulative probability that the portfolio will suffer a loss of more than \$1,000 in a period of one month. It is common to see values of α equal to 0.9, 0.95 and 0.99 in the literature.

VaR_α is a risk measure often used in finance in general and in the portfolio problem in particular. Basak and Shapiro [26] developed an alternative version to the mean-variance approach using VaR_α as the risk measure of the model. Ghaoui et al. [115] assume that the distribution of returns is not known precisely, but is a set of distributions, so implementing an optimization *maxmin* for what they call VaR_α worst case. VaR_α worst case is the largest value of VaR_α given the partial information of the distribution of the returns. Gaivoronski and Pflug [112] look for a way to approach the historical VaR_α by means of a softening function that tries to filter irregularities. They do this in order to decrease the computational complexity of VaR_α. Glasserman et al. [120] concentrate on developing efficient methods to calculate the VaR_α of the portfolios when they present heavy distributions in the tails.

Recent tendency in the related literature indicates a lack of interest of the scientific community mainly due to lack of capabilities of the measure to fulfill the coherence properties mentioned above. Coherence is a desirable characteristic in a risk measure. The coherence guarantees certain properties in the risk measure that are intuitive for the DM. The following example demonstrates that VaR_α is not coherent (Example taken from [203]).

Assume the two portfolios, X_1 and X_2, and their corresponding risk evaluated in 10 states of nature shown in Table 2.3.

Table 2.3: Risk evaluation through VaR

State	X_1	X_2	$X_1 + X_2$
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0
7	0	0	0
8	0	0	0
9	0	1	1
10	1	0	1
VaR	0	0	1

If we calculate VaR_α, (ρ), of the portfolio formed with these two "sub-portfolios" with a percentile of 85%, we have

$$0 = \rho(X_1) + \rho(X_2) < \rho(X_1 + x_2) = 1.$$

So we would violate the axiom of sub-additivity. This means that the VaR_α of a portfolio with a number of sub-portfolios is not necessarily lower than or equal to the sum of the VaR_α of the sub-portfolios that compose it.

On the other hand, in many cases it is useful to know something about the distribution of extreme events. VaR_α provides only the point at which the loss is expected to occur with a predetermined probability, and gives no indication of how likely the loss is if incurred.

Moreover, VaR_α is only compatible with stochastic domination in first order [137].

Conditional value at risk (CVaR) Conditional value at risk ($CVaR$) also called Average value at risk ($AVaR$) and Expected shortfall (ES) is the average of the values that fall beyond the VaR. $CVaR$ is more sensitive than VaR to losses in the distribution tail. It is possible to calculate $CVaR$ of portfolio x by the following parametric method:

$$CVaR_\alpha(x) = -\frac{1}{\alpha}\int_0^\alpha VaR_\gamma(x)d\gamma.$$

$CVaR_\alpha$ is a better risk measure than VaR_α. Not only it overcomes some limitations of VaR, but it also allows one to know how much one can lose on average if the losses exceed the confidence level set by VaR. Aditionally, it satisfies all the axioms of coherent risk measures.

The minimization of $CVaR_\alpha$ also leads to near optimal solutions in terms of VaR since VaR never exceeds $CVaR$. Therefore, portfolios with a low $CVaR_\alpha$ must have a low VaR_α as well [240]. On the other hand, according to Rockafellar and Uryasev [240], when the distribution of return loss is normal, these two measures are equivalent; that is, they provide the same optimal portfolio. However, for very skewed distributions, the portfolios provided by both measures may be different.

Krokhmal et al. [166] developed a model for the optimization of portfolios returns with $CVaR_\alpha$ in the constraints using historical scenarios and carried out a case study optimizing a portfolio formed by S&P100 stocks. The constraints of the portfolio with respect to the $CVaR_\alpha$ were established at different levels. Andersson et al. [12] examined an approach to optimizing credit risk. Optimization is done by minimizing $CVaR_\alpha$ subject to transaction and returns constraints. The risk distribution is generated by Monte Carlo simulations and the optimization is solved by linear programming. Acciaio and Goldammer [2] investigated the case of portfolio optimization with constant proportions over time by minimizing $CVaR_\alpha$. Dhaene et al. [75] investigated portfolio optimization in the classical continuous time style considering $CVaR_\alpha$ and other risk measures based on quantiles. Abad and Iyengar [1] seek to solve the problem of the portfolio with multiple constraints in $CVaR_\alpha$. In the problem they are dealing with, they optimize the weighted sum of the return average and the maximum of a set of spectral risk measures.

Spectral risk measures The value of $CVaR_\alpha$ is the average of the values VaR_α larger than the value VaR_α obtained with a given probability in the tail α. If a weighted average is calculated instead of the simple average of the values VaR_α, it is possible to obtain a family of risk measures known as spectral risk measures (SRMs), where $CVaR$ is a member. The SRMs are related to the coherent measures of risk in the sense that they fulfill the four properties highlighted above. One of the main characteristics of the spectral risk measures is that they relate the level of risk measured with the DM's risk aversion: since the SRMs can be seen as a weighted average of the quantiles of a loss distribution, it is possible to establish the weights of this sum depending on the DM's aversion to risk.

The spectral risk measures are defined as

$$\rho_\kappa(X) = \int_0^1 VaR_p(X)\kappa(p)dp,$$

where $\kappa(p)$, $p \in [0, 1]$, is the weighting function also called risk aversion function and is interpreted in the following way: consider a range $[p_1, p_2]$ of probabilities with length $p_2 - p_1 = \Delta_p$; the weight assigned to this range is approximately $\kappa(p_1)\Delta_p$.

For the measure $\rho_\kappa(X)$ to be considered consistent, the risk aversion function, $\kappa(p)$, must be [235]:

- positive: $\kappa(p) \geq 0, p \in [0,1]$,
- not-increasing: $\kappa(p_1) \geq \kappa(p_2)$, $p_1 \leq p_2$ (larger losses are multiplied by larger weights), and
- normalized: $\int_0^1 \kappa(p)dp = 1$ (the sum of the weights must be 1).

In addition to the four properties of the coherent risk measures mentioned above, the spectral risk measures are

- invariants: for all random portfolio returns X and Y with distribution functions F_X and F_Y, $F_X = F_Y \implies \rho(X) = \rho(Y)$,
- additonally comonotonic[1]: for all comonotonic random variables X y Y, $\rho(X+Y) = \rho(X) + \rho(Y)$.

Adam, Houkari and Laurent [5] evaluate risk from the point of view of risk theory, focusing on spectral risk measures, based on moments and distortion. Subsequently, they apply these ideas in the framework of the hedge fund portfolio.

A spectral risk measure is always a coherent risk measure, but the opposite is not always true. An advantage of the spectral measures is the way in which they can be related to the DM's risk aversion and, in particular, to a utility function through the weights given to the possible returns of the portfolio [63].

Due to the non-increasing property of the risk aversion function in the spectral risk measures, the largest losses, which are to the leftmost part in the tail of the return distribution, are multiplied by a larger weight. Larger losses have greater variability and multiplicate them by the largest weights further increases the variability of the weighted average calculated by the SRMs. Ultimately, this depends on the choice of the DM's risk aversion function and the assumed distribution of the portfolio return. In fact, the assumptions of the distribution of the portfolio X are very important, since they can give rise to SRMs without limits for some risk aversion functions. An infinite risk measure is not informative for the DM. In practice, a bad combination of a statistical model and a risk aversion function can generate problems related to variability in risk estimates. These problems do not exist for $CVaR_\alpha$ because a finite mean return of the portfolio guarantees that $CVaR_\alpha$ is well defined at all the probability levels of the tail. The problem for spectral risk measures arises from the non-increasing property of the risk aversion function. Higher losses are multiplied by larger weights, which can result in an unlimited weighted average [235].

Quantiles Greco et al. [126] recently proposed an approach to the portfolio selection based on the consideration of some significant quantiles of the probability distribution of the returns as criteria to be maximized.

[1] Two random variables X and Y are comonotonic if $\forall(\omega_1, \omega_2) \in \mathbb{R}^2, (X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$

Let X be the random variable representing the return of a portfolio, a quantile q of X is the value a such that $\mathbb{P}(X \geq a) = q$. That is, the quantile q represents the value of the variable X that marks a limit so that a proportion of q values of the population is greater than or equal to a.

These quantiles are easily understandable even for a DM without sophisticated financial preparation, since each quantile simply states the minimum return that a portfolio would give with a corresponding probability value. While the Markowitz approach solves the problem of the portfolio trough a bi-objective problem in terms of mean and variance, the approach of Greco et al. [126] solves the same problem by transforming it into a multicriteria decision problem in which a set of quantiles in relation to some probability values have to be maximized.

To use these quantiles, Greco et al. [126] propose using an interactive multiobjective optimization combined with the Dominance Rough Set Approach (DRSA) as the main method to solve the portfolio selection problem that consists of two phases: the calculation phase and the dialogue phase. In the calculation phase, a sample of the feasible portfolios is calculated and presented to the DM. In the dialogue phase, the DM indicates which portfolios are attractive to him/her; this binary classification of the portfolios in "good" and "other" is input preference information to be analyzed using Rough Sets, which produces decision rules related to the conditions of the quantiles with the intention to support "good" portfolios.

Unfortunately, the number of criteria to be optimized using this approach might exceed the cognitive limitations of the decision maker; above all, when many objectives are stablished by the decision maker.

2.1.4 Stock portfolios

Few human activities have been so exhaustively studied during the past century as the buying and selling of the so-called corporate stocks. It has attracted the attention of researchers from many areas due to both the interesting theoretical challenges and the wonderful possible rewards. In the course of years of stock market study, two quite distinct schools of thought have arisen, providing two radically different methods of arriving at the answers to the trader's problem of what and when. One of these is commonly referred to as the *fundamental* analysis, and the other as the *technical* analysis.

2.1.4.1 Fundamental Analysis

The stock market investor based on the fundamental analysis depends on statistics. She/He examines the auditors' reports, the profit-and-loss statements, the quarterly balance sheets, the dividend records, and the policies of the companies whose stocks she/he has under observation. Such investor analyzes sales data, managerial ability, plant capacity, and the competition. She/He turns to bank and treasury reports, production indexes, price statistics, and crop forecasts, to gauge the state of business in general, and reads the daily news carefully to arrive at an estimate of future business conditions. Taking all these into account, the investor evaluates her/his stock; if it is selling currently below her/his appraisal, she/he regards it as a buy.

The information provided by the fundamental analysis is mainly used in the literature to select competitive stocks. Although this information may be qualitative, it is often generated in the form of ratios of numerical values taken from the financial statements of the companies. Many works in the literature usually aggregate these indicators in a global evaluation index through a subjective process that may depend on the DM decision policy (see e.g., Ref. [287]). Such aggregation is a problem *per se.*

It is well known that the fundamental analysis can be different for companies with different business activities; for example, for financial and non-financial companies [194]. Therefore, an exploration of the most convenient indicators should be used when the fundamental analysis is exploited (e.g., [287]). Some fundamental indicators that can be used for trans-business companies are shown in Table 2.4 (cf. Refs. [194,287]).

Table 2.4: Fundamental indicators that can be used for companies with different business activities.

Indicator	Name	Definition
if_1	Return on assets	Earnings before interest and taxes divided by total assets.
if_2	Return on equity	Net income divided by shareholders equity.
if_3	Earnings Per Share	Net income minus dividends on preferred stocks all divided by average outstanding shares.
if_4	Dividend yield	Annual dividends per share divided by price per share.
if_5	Price on earnings	Market value per share divided by earnings per share
if_6	Price on book	Stock price divided by all total assets minus intangible assets and liabilities.
if_7	Price on sales	Share price divided by revenue per share.
if_8	Price on cash Flow	Share price divided by cash flow per share

2.1.4.2 Technical Analysis

The technical analysis studies the market patterns, demand and supply of stocks [3]. It consists of using price data to create rules and exploit them financially by selecting stocks in accordance with them. If the rule associated with a technical indicator shows that the price of a stock is likely to rise, the DM should buy now expecting to sell later at a higher price, thus increasing the return of the portfolio.

Some of the most frequently mentioned technical indicators reported in the literature are (cf. e.g., Refs. [19,124,188]): Exponential Moving Average (EMA), Double Crossover (DC), Rate of Change (ROC), Relative Strength Index (RSI), Moving Average Convergence/Divergence (MACD), On Balance Volume (OBV), Bollinger Band (BB), and True Strength Index (TSI). Let us now describe the rule associated with each of these indicators.

The EMA is one of the simplest technical indicators, where higher weights are assigned to the most recent data. The EMA for the ith stock in period t, EMA_t^i, is defined as (cf., Ref. [188]):

$$EMA_t^i(ws) = [p_t^i - EMA_{t-1}^i(ws)] \cdot w + EMA_{t-1}^i(ws),$$

where ws is the length of the sliding window of the exponential moving average, $w = \frac{2}{ws+1}$, and the initial EMA (i.e., when $t = ws$) is calculated as the average of the previous ws periods. A value of $ws = 12$ is commonly used (e.g., Gorgulho et al., 2011). The rule associated with this indicator states that if the price line crosses above the EMA line, then the stock should be supported. Formally (cf. Ref. [124]):

$$it_1^i = \begin{cases} 1 & EMA_t^i(12) > p_t^i \wedge EMA_{t-1}^i(12) < p_{t-1}^i, \\ 0 & \text{otherwise.} \end{cases}$$

Where $\wedge$ is the conjunction operator.

The DC uses two moving averages (normally, a short one and a large one) and produces a signal when the shorter crosses above the larger. Normally, a window size of five periods is used for the short line while the large line uses a window size of 20 periods. Hence, the signalization rule for this indicator is as follows:

$$it_2^i = \begin{cases} 1 & EMA_t^i(5) > EMA_t^i(20) \wedge EMA_{t-1}^i(5) < EMA_{t-1}^i(20), \\ 0 & \text{otherwise.} \end{cases}$$

The ROC represents the proportional difference between the current price of the ith stock and the price h periods ago (cf., Ref. [19]): $ROC_t^i = \frac{p_t^i - p_{t-h}^i}{p_{t-h}^i}$. Positive values in this indicator are desirable. A value $h = 13$ is commonly accepted [124]:

$$it_3^i = \begin{cases} 1 & ROC_t^i(13) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The RSI is a momentum oscillator conceived to measure the relative conditions of the stock in the market with respect to its overbought/oversold condition. The RSI value for the ith stock in period t is defined as (cf. Refs. [188,286]):

$$RSI_t^i(d) = 1 - \frac{1}{1 + RS_t^i(d)},$$

where $RS_t^i(d) = \frac{\sum_{j=1}^{d} U_{t-j+1}^i}{d} / \frac{\sum_{j=1}^{d} D_{t-j+1}^i}{d}$; $\sum_{j=1}^{d} U_{t-j+1}^j$ is the sum of the positive returns of stock i during d periods before the period t, and $\sum_{j=1}^{d} D_{t-j+1}^i$ is the same sum but with negative returns. It is widely accepted that $d = 14$ (e.g., Refs. [124,286]). This indicator suggests that the ith stock should be supported in period t if the RSI_t^i value crosses above 30% and the current price is higher than the price of the previous period (see Refs. [124,188]). Formally:

$$it_4^i = \begin{cases} 1 & RSI_t^i(d) > 0.3 \wedge RSI_{t-1}^i(d) < 0.3 \wedge p_t^i > p_{t-1}^i, \\ 0 & \text{otherwise.} \end{cases}$$

The MACD is a combination of EMAs that validates the "convenience" of acquiring a stock through the comparison with a signaling function. The most common configuration of this indicator uses two EMAs with 12 and 26 historical periods, $EMA(12)$ and $EMA(26)$, to create the $MACD(12, 26)$ [19,124,188]. The MACD for the stock i in the period t is defined as $MACD_t^i(12, 26) = EMA_t^i(12) - EMA_t^i(26)$. The literature usually traces another moving average that does not depend on the price of the stocks but depends on the $MACD_t^i$ indicator. This new moving average, MM_t^i, is used to create a signal line of momentum about the movement of the prices. The signal line is created as a nine-period EMA of the $MACD_t^i$. The common strategy associated with this indicator states that when the value of $MACD_t^i(12, 26)$ crosses above $MM_t^i(9)$, then there is evidence that the stock will increase in price and that it is advisable to invest in it in the current period. Thus, we will consider that this indicator suggests support for the ith stock if $it_5^i = 1$, where (cf., Refs. [19,124]):

$$it_5^i = \begin{cases} 1 & MACD_t^i(12, 26) > MM_t^i(9) \wedge MACD_{t-1}^i(12, 26) < MM_{t-1}^i(9), \\ 0 & \text{otherwise.} \end{cases}$$

The OBV assumes that a rising volume might precede a rise on the stock's price and is calculated as [124]:

$$OBV_t^i = \begin{cases} OBV_{t-1}^i + vol_t^i & p_t^i > p_{t-1}^i, \\ OBV_{t-1}^i - vol_t^i & p_t^i < p_{t-1}^i, \\ OBV_{t-1}^i & \text{otherwise.} \end{cases}$$

Where vol_t^i is the volume (number of shares traded) of the ith stock in period t.

The OBV indicates that the ith stock should be supported if the value OBV_t^i is rising simultaneously with price (indicating a clear up trend). Formally:

$$it_6^i = \begin{cases} 1 & OBV_t^i > OBV_{t-1}^i \wedge p_t^i > p_{t-1}^i, \\ 0 & \text{otherwise.} \end{cases}$$

The BB is a strategy of election with strong positive net results [188]. It is a volatility indicator represented by the bands generated from an l-day price moving average minus 2 standard deviations of price changes over the same l-periods time span:

$$MA_t^i(l) = \sum_{j=1}^{l} \frac{p_{t-j+1}}{l}.$$

$$LB_t^i(l) = MA_t^i(l) - 2\sigma_t^i(l).$$

Where $\sigma_t^i(l)$ is the standard deviation of price changes of stock i for the period t and its previous $l-1$ periods.

The rule associated with the BB states that the ith stock should be supported if in period t its price is simultaneously above $LB_t^i(l)$ and below $MA_t^i(l)$ (to avoid false triggering; cf. Ref. [188]):

$$it_7^i = \begin{cases} 1 & MA_t^i(l) > p_t^i > LB_t^i(l), \\ 0 & \text{otherwise.} \end{cases}$$

The TSI is double smoothed with two moving averages to show the trend and specifying, at the same time, the overbought and oversold conditions Ref. [124]. It can be defined as:

$$TSI_t^i(r,s) = 100 \times \frac{EMA_t^i(S) \text{ of } (EMA_t^i(r) \text{ of } diff^i)}{EMA_t^i(S) \text{ of } (EMA_t^i(r) \text{ of } |diff^i|)},$$

Where $diff^i$ is the momentum line which calculates the difference between the current price and the price observed on the previous period, that is $diff^i = p_j^i - p_{j-1}^i$ for a given period j.

The literature often uses an EMA of the TSI as trigger: $Trigger_t^i(m) = EMA_t^i(m)$ of TSI_t^i and an *oversold region*. Such region indicates that the stock's price is lower than it should be, and usually is located in the value -25 of TSI_t^i. The rule associated with the TSI states that the ith stock should be supported if the TSI_t^i crosses above $Trigger_t^i(m)$ on the oversold region, that is:

$$it_8^i = \begin{cases} 1 & TSI_t^i(r,s) > Trigger_t^i(m) \wedge TSI_{t-1}^i(r,s) < Trigger_{t-1}^i(m) \wedge \\ & Trigger_t^i(m) \leq -0.25, \\ 0 & \text{otherwise.} \end{cases}$$

2.1.4.3 Stock indexes as benchmarks

A market index is a way of measuring the value of a section of the stock market. More specifically, it is an aggregated value that is produced by combining various stocks of the market section. Since the market indexes arise from a mathematical construction, it is not possible to invest directly in them. However, it is a tool used by investors to describe the market and compare the performance of the portfolios.

There are two streams of thought to create stocks portfolios [124,189,267]: passive management and active management. The first stream states that "the past movement or direction of return of a stock, or of the market in general, cannot be used to predict its future movement" [190]. And that in trying, the DM spends resources that in the long run can be rather detrimental. As a result, "there has been an accelerating trend in recent decades to invest in passively managed investment funds based on market indexes, known as index funds" [267]. The goal of these funds is not to outperform the corresponding index, but instead to track the index as closely as possible by buying each of the stocks in the it in amounts equal to the weights in the index itself. As index funds try to replicate index holdings, they eliminate the need -and therefore many costs- for the research involved in active management. This makes indexes one of the main benchmarks in the selection of stock portfolios (see e.g., [124,181,287]).

On the other hand, active management depends on analytical research, estimations, and the judgment and experience of the decision maker to form portfolios. The objective of active management is to outperform a reference index [124]. It can be done through the incorporation of decision-maker preferences, estimation of portfolio return (e.g., expected return), measurement of risk of not obtaining the return estimated (e.g., standard deviation), and the purchase of undervalued stocks (e.g., through financial indicators).

A highly important market index is the Dow Jones Industrial Average, DJIA. The DJIA index contains the stocks of 30 of the largest companies in the United States.

Following Soe and Poirier [267], the main contraindication of using market indexes as benchmarks is that the profitability of portfolios is often compared to popular indexes such as DJIA, regardless of portfolio size or classification of its stocks. Most investors expect to reach or exceed the yields of these indexes over time. The problem with this expectation is that they are at a disadvantage because they are not "comparing apples to apples". That is, there is no guarantee that the characteristics of the stocks in the portfolio coincide with the characteristics of the stocks contained in the index. We avoid this trap by incorporating into the portfolio only the stocks of the index being considered as benchmark.

2.2 Interval Theory

The so-called Interval Analysis Theory was originated independently by Sunaga [274] and Moore [213]. The principal concept of Interval Theory is the *interval number*. Such a number represents a numerical quantity whose exact value is unknown. Given this imperfect knowledge about the quantity, a range of numbers is used to encompass all the possible values that the quantity could obtain. In this way, an interval number stands for an indeterminate number that takes its possible value within a set of numbers. Let us consider the quantity ι whose real value lies between bounds i^- and i^+. The interval number for such quantity is set then as $I = [i^-, i^+]$. Any $r \in [i^-, i^+]$ is called a realization of I. We can also translate a real number, q, into an interval number as $[q, q]$.

In what follows, let us look at the basic operations of interval numbers. Given the interval numbers $I = [i^-, i^+]$ and $J = [j^-, j^+]$, the following equations represent the addition, subtraction, multiplication and division, of I and J, respectively.

$$
\begin{aligned}
I + J &= [i^- + j^-, i^+ + j^+],\\
I - J &= [i^- - j^+, i^+ - j^-],\\
I \times J &= [\min\{i^-j^-, i^-j^+, i^+j^-, i^+j^+\}, \max\{i^-j^-, i^-j^+, i^+j^-, i^+j^+\}],\\
I \div J &= [i^-, i^+] \times [\frac{1}{j^-}, \frac{1}{j^+}].
\end{aligned}
$$

More recently, Shi et al. [260] proposed a way to determine the order of interval numbers. For instance, suppose we want to determine the order of $I = [i^-, i^+]$ and $J = [j^-, j^+]$. First, we need to find the possibility of I being greater than or equal to J. The possibility function proposed in Ref. [260] is given by

$$
p(I \geq J) = \begin{cases} 1 & \text{if } p_{\{IJ\}} > 1,\\ p_{\{IJ\}} & \text{if } 0 \leq p_{\{IJ\}} \leq 1,\\ 0 & \text{if } p_{\{IJ\}} < 1. \end{cases} \tag{2.2.4}
$$

Where $p_{\{IJ\}} = \frac{i^+ - j^-}{(i^+ - i^-) + (j^+ - j^-)}$.

Furthermore, if $i^+ = i^-$ and $j^+ = j^-$, then

$$
p(I \geq J) = \begin{cases} 1 & \text{if } I \geq J,\\ 0 & \text{otherwise.} \end{cases}
$$

Let i and j be two currently undetermined realizations from I and J, respectively; $p(I \geq J)$ can be interpreted as a degree of credibility of the statement "once both realizations are determined, i will be greater than or equal to j". This helps the DM to ensure the robustness of $I \geq J$, that is, to have a strong belief on I being not less than J when they are instanced as real numbers [97].

It is easily proved (see [97]) that Equation (2.2.4) fulfills some interesting properties: let $I = [i^-, i^+]$, $J = [j^-, j^+]$, and $K = [k^-, k^+]$, then:

- $p(I \geq J) = 0$ if $j^- > i^+$ and $p(I \geq J) = 1$ if $j^+ \leq i^-$.
- If $j^- = i^-$ and $j^+ = i^+$, it is said that I is equal to J, denoted as $I = J$. Then $p(I \geq J) = 0.5$.
- If $i^- > j^+$, it is said that I is strictly greater than J, denoted as $I > J$. Then $p(I \geq J) = 1$.
- If $i^+ < j^-$, it is said that I is strictly lower than J, denoted as $I < J$. Then $p(I \geq J) = 0$.
- When $p(I \geq J) > 0.5$, it is said that I is strictly greater than J, denoted as $I > J$.
- When $p(I \geq J) < 0.5$, it is said that I is strictly lower than J, denoted as $I < J$.
- If $p(I \geq J) = \alpha_1 \geq 0.5$ and $p(J \geq K) = \alpha_2 \geq 0.5$ then $p(I \geq K) \geq \min\{\alpha_1, \alpha_2\}$.
- If $\hat{i}$ and $\hat{j}$ are respectively the middle points of the confidence intervals I and J, we have $I > J$ if and only if $\hat{i} > \hat{j}$ and $I = J$ if and only if $\hat{i} = \hat{j}$.
- If $p(I \geq J) = \alpha > 0.5$ then $p(J \geq I) = 1 - \alpha < 0.5$.

Finally, we define the concept of a maximum among a set of interval numbers as follows. Let $\mathscr{B}$ be a set of interval numbers, $b^* \in \mathscr{B}$ is the maximum of $\mathscr{B}$, denoted by $\max\{\mathscr{B}\}$, if and only if $p(b^* \geq b) \geq 0.5$ for all $b \in \mathscr{B}$.

2.3 Multicriteria decision process

Making a decision consists in accepting or rejecting a potential action and/or performing it in a certain way. Making a comprehensive decision is rarely a straightforward process; rather, there usually are confrontations in successive interactions among stakeholders. It is the playing out of these confrontations and interactions, under the various compensating and amplifying effects of the stakeholders' preference systems, that makes up what we shall call the decision process [244].

There is a special type of decision called *strategic decision* that can be defined as the one that is "important, in terms of the actions taken, the resources committed, or the precedents set" [209]. High levels of uncertainty and negotiations with the stakeholders often characterize strategic decisions. These stakeholders are *actors*, individuals or entities who have an important interest in the decision and will intervene to directly affect it through the value systems which they possess. One of these actors is the so-called decision maker (DM) who is the person or the set of persons for whom or in the name of whom decision aiding effort is provided; the DM is responsible of the decision's consequences. Other important actor in the decision process is the so-called analyst. The analyst role is to explain, justify, and recommend during the decision process regardless of her/his own preference system but surely influencing the decision.

The uncertainty involved in making a strategic decision is frequently due to the complexity of real-world problems, which in turn is due to the dimensionality of the given problem (e.g., the number of available options and actions) and the nature of the available data which are often imprecise. Particularly, an important implication of addressing real-world decision-making problems, is the multi-dimensionality character that requires the consideration of multiple conflicting points of view. Therefore, the decision process should explore the conflicting nature of the criteria, their trade-offs, the decision makers' objectives, and the way that the decision model can deal with all these. Moreover, such decision model must always take into account the preferences of the decision maker. The idea of the optimal decision (optimal solution to a decision problem) is abandoned for the notion of the "satisfaction of the decision maker".

We will refer to strategic decisions simply as *decisions* during the rest of this document.

2.3.1 Basic definitions

We use in this document the term *potential action* to label what constitutes the object of the decision, something through which the decision will materialize. As opposed to the term *alternative*, potential actions are not necessarily stable (i.e., potential actions can evolve during the decision-making process) and more than one potential action can be jointly put into operation. We use $\mathscr{A}$ to designate the set of potential actions to consider in a decision-making process.

Each potential action $a \in \mathscr{A}$ can be evaluated through a criterion $g_j(a)$ reflecting, possibly with a certain fuzziness, the preferences of one or several actors on a given point of view in such a way that $a \in \mathscr{A}$ is better than $b \in \mathscr{A}$ according to this point of view, without loss of generality, if and only if $g_j(a) > g_j(b)$. When the actors of the decision process require to assess the potential actions according to several points of view, a family of criteria $\mathscr{F} = \{g_1, g_2, \cdots, g_k\}$ must be used. Such family of criteria must follow the requirements that lead to the definitions of exhaustiveness, cohesiveness, and non-redundancy that characterize the concept of a coherent criterion family (see Ref. [245]). The elements of the domain of a criterion are called scale values, each of them can be characterized for example by a number or a verbal declaration. To compare two actions according to a criterion, we compare the two values used to evaluate them. This leads to distinguish between several types of scales, in particular the ordinal and the cardinal scales. The former is such that the space between two values has no clear meaning in terms of the difference in preferences. The cardinal scales (interval, ratio, absolute) are numerical scales that give meaning to value differences, proportions and absolute numbers, respectively. Finally, we consider here each $g_j(\cdot)$ as a real-valued function.

2.3.2 Multicriteria decision problems

An multicriteria decision problem consists in elaborate either a mathematical model allowing us to compare potential actions in a comprehensive way or a procedure helping to reflect upon and to progress in the formulation of comprehensive comparisons between potential actions [244]. With respect to the objectives of the decision process, three different perspectives corresponding to the following three problem statements can be identified [242]:

- (Pα) Identify the best potential action or select a minimal set of the best potential action.
- (Pβ) The assignment of each potential action to an appropriate pre-defined class or category.
- (Pγ) Create a ranking of the potential actions which seems to be the most satisfactory according to a total or partial pre-order.

When several criteria are considered, generally there is not a common optimal point but the best compromise solution must be found according to the DM's preference system. Multicriteria problems are more complicated to be solved than their mono-criterion counterpart from the DM's subjectivity point of view. The decision set of the former can be is expressed by variables that must fulfill certain mathematical constraints, which separate what is feasible from what is not (the so-called mathematical programming problems) or by describing a list of potential actions (the so-called discrete decision problems).

Decision problems can be classified according to the level of knowledge about the consequences that making a decision causes: those in which the consequences of the decisions are supposedly known with certainty, and the decision problems under risk or uncertainty that correspond to those with results that are known with little or no precision. The decision under conditions of certainty supposes that each of the potential actions leads with perfect certainty to the DM to a well-defined consequence. The DM has a complete knowledge of everything that she/he considers relevant to his problem. The decision under conditions of "pure" certainty is probably an ideal. In practice there is always a level of imprecision. Sometimes we can neglect or handle imprecision with sensitivity analysis techniques, performing a treatment similar to the complete certainty model. Decision situations under imprecise conditions are more realistic. There are situations in which the probabilities of the states of nature are known, and others in which no information about the states of nature appears. The situation of the first type is called under risk; the second is called under complete uncertainty [109]. There are also situations in which it is not possible to model the imprecision in a probabilistic way, at least not without an appreciable level of arbitrariness [242,244].

2.3.3 Main schools of decision

Two main approaches to provide aid in decision-making processes can be highlighted (cf. [224]): i) the normative approach or value function model, which takes as a fundamental premise the DM's rationality and consistency and whose primary objective is building value or utility functions; and ii) the relational approach, which, based on certain preferential information, derives a preference statement between pairs of actions. Let us now describe these approaches.

Normative approach

A mathematical modeling of preferences developed on an axiomatic basis is the main aspect of the value function model. This is carried out assuming DM's consistency (i.e., ideal behavior), which is described as: a) unlimited, conferring capabilities to perceive and evaluate differences between potential actions that would be descriptively imperceptible; b) unrestricted, in the sense that the DM can not reject the task of making preference judgments between two potential actions.

A well-defined set $\mathscr{A}$ of feasible alternatives is considered by this approach. The definition of this set can take two forms: i) in the analytical form, the alternative $a \in \mathscr{A}$ is a vector $a = (x_1, x_2, \cdots, x_n)$ and is considered a well defined set of constraints that leads the set $\mathscr{A}$ to be part of $\mathbb{R}^n$; ii) in the enumerative form, $\mathscr{A}$ is described by a list of alternatives a and it is not considered any kind of relationship with a set of constraints.

Considering $a, b \in \mathscr{A}$, the DM is able to select (unambiguously) one and only one of the following possibilities:

- $a \succ b$: a is strictly preferred to b. It implies that the DM prefers alternative a to alternative b; in other words, there are clear and positive reasons that the DM would be disappointed if he were forced to select alternative b instead of alternative a.
- $b \succ a$: b is strictly preferred to a.
- $a \sim b$: a is indifferent to b. It means that the DM is indifferent between alternatives a and b; In other words, there are clear and positive reasons that the DM would not be disappointed if he were forced to select either of the two alternatives.

These relationships must fulfill the following properties:
Transitivity. Let $a, b, c \in \mathscr{A}$, if the DM states that $a \succ b$ and $b \succ c$, then she/he must also affirm that $a \succ c$.
Asymmetry. Let $a, b, c \in \mathscr{A}$, if the DM states that $a \succ b$, then she/he should not claim that $b \succ a$.
Transitivity. Let $a, b, c \in \mathscr{A}$, if the DM states that $a \sim b$ and $b \sim c$, then she/he must also affirm that $a \sim c$.

Reflexivity. For every option $a \in \mathscr{A}$ it happens that $a \sim a$.

Symmetry. Let $a, b \in \mathscr{A}$, if the DM states that $a \sim b$, then she/he must also affirm that $b \sim a$.

Transitivity as a whole. Let $a, b, c \in \mathscr{A}$, if the DM states that $a \sim b$ and $b \succ c$, then she/he must also affirm that $a \succ c$; and if she/he say that $a \succ b$ and $b \sim c$, then she/he must also affirm that $a \succ c$.

What has been described up to now does not specify the existence of risk on the levels of performance of the alternatives. The best known extension of the normative approach to risk treatment is based on the von-Neumann and Morgenstern Utility Theory combined with de Finetti's subjectivist approach to probability (see [25]). As an alternative of decision, certain structures called lotteries are considered, which describe the consequence and its probability. The axioms of the value function, together with others related to lotteries, show that there is a particular value function, called utility, that models the DM risk attitude, so that the most appropriate decision alternative is the one that maximizes expected utility [109].

The popular representation of a lottery is:

$$l = <p_1, x_1; p_2, x_2; \cdots ; p_r, x_r>$$

where $p_i \geq 0$ represents the probability of winning the prize x_i, $i = 1, 2, \cdots, r$ and $\sum p_i = 1$.

It is also common to see compound lotteries, whose prizes are to win entry to another lottery:

$$l = <q_1, l_1; q_2, l_2; \cdots ; q_s, l_s>$$

where $q_i \geq 0$ represents the probability of winning the lottery entry l_i, $i = 1, 2, \cdots, s$ and $\sum q_i = 1$.

Let X be the set of direct rewards (not lotteries) that can be obtained from a lottery; L the set of lotteries (simple or compound) on which the DM makes the decision; R_L the set of possible results that can be obtained from lotteries; and, for convenience and without loss of generality, let's set the prizes in such a way that $x_1 \succeq x_2 \succeq \cdots \succeq x_r$. Let's also define lotteries described as "reference lotteries" in the following way:

$$<p, x_1; 0, x_2; \cdots ; 0, x_{r-1}; (1-p), x_r>$$

where the DM gets the best prize (x_1) with a probability p and the worst prize (x_r) with a probability $p - 1$. Represent this type of lottery as $x_1 p x_r$. Suppose the existence of a utility function $u : \mathbb{R} \longrightarrow \mathbb{R}$ defined on $X = \{x_1, x_2, \cdots, x_r\}$, such that:

$$x_j \succeq x_k \Longleftrightarrow u(x_j) \geq u(x_k) \forall x_j, x_k \in X,$$

and

$$<p_1, x_1; p_2, x_2; \cdots ; p_r, x_r> \succeq <p'_1, x_1; p'_2, x_2; \cdots ; p'_r, x_r> \Longleftrightarrow \sum p_i u(x_j) \geq \sum p'_i u(x_j).$$

Let $w_1, w_2 \in R_L$. It is said that there is comparability in R_L since at least one of the following propositions is true:

$$w_1 \succeq w_2;$$

$$w_2 \geq w_1.$$

Von-Neumann and Morgenstern formalized a rational decision paradigm under risk defined by the following set of axioms.

Axiom 1 (weak order). For each pair of lotteries, $\geq$ is a weak order (comparable and transitive relationship).

Axiom 2 (not triviality). $x_1 > x_r$.

Axiom 3 (reduction of lotteries composed of a simple lottery). Let

$l = <q_1, l_1; q_2, l_2; \cdots; q_s, l_s>$, where $l_j = <p_{j1}, x_1; p_{j2}, x_2; \cdots; p_{jr}, x_r>; j = 1, 2, \cdots, s$. If $l' = <p_1, x_1; p_2, x_2; \cdots; p_r, x_r>$, where $p_i = q_1 p_{1i} + q_2 p_{2i} + \cdots + q_s p_{si}; i = 1, 2, \cdots, r$, then the DM must affirm that $l \sim l'$.

Axiom 4 (substitutionality). Let $a_i, b, l, l' \in R_L$. Let

$l = <q_1, a_1; q_2, a_2; \cdots; q_i, a_i; \cdots; q_r, l_r>$ and $l' = <q_1, a_1; q_2, a_2; \cdots; q_i, b; \cdots; q_r, l_r>$. If $a_i \sim b$, then the DM must state that $l \sim l'$.

Axiom 5 (the reference experiment). The DM can express preferences about lotteries of the form $x_1 p x_r \in L$ $\forall p, 0 \leq p \leq 1$.

Axiom 6 (monotony). $x_1 p x_r \geq x_1 p' x_r \iff p \geq p'$.

Axiom 7 (continuity). $\forall x_i \in X\ \exists u_i, 0 \leq u_i \leq 1$, such that $x_i \sim x_1 u_i x_r$.

A relevant concept in the theory of the decision in conditions of risk considered by the normative approach is the concept of Certainty Equivalent that corresponds to the prize that the DM would consider indifferent to enter the lottery. That is, if the DM is presented with the option of entering the lottery or receiving the Certainty Equivalent, she/he would be indifferent to one or the other option. More formally, if l is a lottery and u a utility function, the certainty equivalent of l is x_c such that $u(x_c) = E[u(l)]$.

$E[u(l)]$ can be obtained only when the DM plays l a sufficient amount of times. However, this situation is difficult to occur, so it is necessary to take into account the DM's attitude facing risk. This attitude is subjective and can be classified as:

- averse to risk when $E[u(l)] < u[E(l)]$ and $x_c < E(l)$.
- risk-taker when $E[u(l)] > u[E(l)]$ and $x_c > E(l)$.
- neutral at risk when $E[u(l)] = u[E(l)]$ and $x_c = E(l)$.

In order to carry out a correct decision support task in risk conditions, it is necessary to carry out a procedure in accordance with the attitude of the DM.

Relational approach

The normative approach has successfully achieved the formality of a theory based on axioms, concentrating on the creation of formal bases to model the preferences of the DM, however, it has disadvantages due to the rigidity that mathematical formalism adds to real phenomena of human behavior, modeling only ideal situations with a proper concept of rationality.

The relational approach tries to avoid the limitations of the normative approach through a flexible model that does not demand DM behaviors that are rigid and, sometimes, far from reality. This approach implements a methodology called multicriteria decision aid (MCDA) that can be defined as follows (see [244]): MCDA is the activity that, by using explicit models, but not necessarily formalized in a complete way, helps to obtain elements of answers to the questions of those interested in a decision process. These elements work to clarify the decision or simply favor a behavior that increases the consistency between the evolution of the process and the objectives and value system of the stakeholders. MCDA considers premises different to the ones considered by the normative approach. In the former, the decision problem is based on formulating a judgment between two potential actions whose consequences may not be known with precision.

According to Roy, in Ref. [244], under the conception of MCDA the analyst does not need to accept any of the following postulates:

- Postulate of the DM optimum. In the context of the decision studied, there is at least one optimal decision, or, in other words, there is a decision for which it is possible (if sufficient time and means are available) to establish objectively that there are no strictly better decisions regarding the system of DM preferences.
- Postulate of the reality about the decision context. The main aspects of the reality on which the decision aiding is based (particularly the preferences of the DM) are related to knowledge objects that can be seen as data (i.e., they exist outside the form in which they are modeled); these objects can also be seen as sufficiently stable during the time in which the questions are asked, so that it is possible to query the exact state or exact value (deterministic or stochastic) of given characteristics that are judged to accurately portray an aspect of reality.

Furthermore, there are several reasons to avoid the dilemma of having to choose between strict preference and indifference:

a) Sometimes it is not possible to discriminate between $\succ$ and $\sim$. The information available may be incomplete or too subjective. Implying indifference due to the lack of information is equivalent to taking arbitrary risks.

b) The DM preferences may not be well defined. For example, that always happens (though not only) when the DM is a group entity.

c) Sometimes it is not possible to determine the preferences of the DM, since she/he may be inaccessible to the analyst, or be an ill-defined entity.

d) Sometimes the DM does not want to discriminate between preference and indifference, because of insecurity, lack of information, imprecision or important contradictions between the attributes of the decision.

The objective of the MCDA is to provide the DM with a set of scientific bases that serve as a tool to carry out a judgment procedure that has as its outcome a solution that satisfies her/him. According to the situation, this methodology can contribute to [101]:

- analyze the context of decision-making by identifying the actors, the possibilities of action, their consequences, $\cdots$;
- organize and/or structure how the decision-making process is developed to increase the coherence between, on the one hand, the values underlying the objectives and goals and, on the other, the decision made;
- to achieve the cooperation of the actors when proposing keys for a better mutual understanding and a favorable working framework to debate;
- elaborate recommendations using the results taken from computational models and procedures conceived within the framework of a working hypothesis;
- participate in the legitimation of the final decision.

2.3.4 Interval-based outranking approach

Fernandez et al. [97] recently proposed a novel approach called *interval-based outranking* that generalizes the classical outranking method. The classical outranking approach assumes that the DM's decision policy can be modeled by a preference model that contains punctual values of criterion weights, veto thresholds and criterion scores; imperfect knowledge on criterion performance is modeled by using indifference and preference thresholds. The interval-based outranking approach assumes that these preference parameters and criterion scores are imperfectly known; imperfect knowledge is modeled through the representation of parameters as interval numbers.

If we assume that $\mathscr{A} = \{x, y, \cdots\}$ is the set of potential actions and $\mathscr{F} = \{g_1, g_2, \cdots, g_k\}$ is a coherent family of criteria (in the sense expressed in Ref. [46]) where, without loss of generality, it is assumed that all criteria are to be maximized, then some of the parameters used by the interval-based outranking are the following. (Note the definition of the parameters as interval numbers.)

- $g_j(x) = [g_j^-(x), g_j^+(x)]$, the performance of $x \in \mathscr{A}$ in criterion g_j;
- $w_j = [w_j^-, w_j^+]$, the weight of criterion g_j;

- $v_j = [v_j^-, v_j^+]$, the veto threshold of criterion g_j; and
- $\lambda = [\lambda^-, \lambda^+]$ reflects a threshold for a sufficient strength of the concordance coalition.

Where $x \in \mathscr{A}$ and $j = 1, 2, \cdots, k$. Since the imperfect knowledge on the criterion performances is represented through intervals, no preference and indifference thresholds are used in Ref. [97].

Through the previous parameters, the interval-based outranking builds a likelihood index between pairs $(x, y) \in \mathscr{A} \times \mathscr{A}$, $\beta(x, y) \in [0, 1]$, of the assertion "x is at least as good as y", xSy. This approach also uses a cutting level, β_0, such that $xSy \iff \beta(x, y) \geq \beta_0$. Below we present a description of the method proposed in Ref. [97].

The marginal likelihood index, $\alpha_j(x, y)$, on solution x being at least as good as solution y with respect to criterion g_j is calculated as

$$\alpha_j(x, y) = p(g_j(x) \geq g_j(y)).$$

Where $p(\cdot)$ is the possibility function described in Equation (2.2.4). If the existence of a likelihood threshold δ_j for each criterion g_j is assumed, then the set of all criteria for which $\alpha_j(x, y) \geq \delta_j$ is called "*concordance coalition* with the assertion xSy" and is denoted by $C(xS_\delta y)$. This concordance coalition is associated with an index $\delta = \min\{\alpha_j(x, y) : g_j \in C(xS_\delta y)\}$. δ is the likelihood that all criteria in the concordance coalition are actually in agreement with xSy. (Recall that the performance of the solutions and the values in the set of preference parameters are imperfectly known, so it is not possible to guarantee a total concordance of the criteria.) Criteria that are not in $C(xS_\delta y)$ compose the discordance coalition, $D(xS_\delta y)$. All this is formalized as

$$g_j \in C(xS_\delta y) \iff \alpha_j(x, y) \geq \delta, \text{ and}$$

$$D(xS_\delta y) = \mathscr{F} - C(xS_\delta y).$$

The imprecision in the definition of the parameters makes it impossible to guarantee $\sum w_j = 1$, as in the classical outranking method. Any realization of the weights is valid only if that condition is fulfilled. So, it must be ensured, at least, that it can be fulfilled. The following two feasibility constraints are established with this purpose.

$$\sum_{j=1}^{k} w_j^- \leq 1 \tag{2.3.5}$$

$$\sum_{j=1}^{k} w_j^+ \geq 1 \tag{2.3.6}$$

The concordance index of the statement "x is at least as good as y", $c(x, y) = [c^-(x, y), c^+(x, y)]$, is defined as follows. First, it is intuitive to assume that

$$c^-(x, y) = \sum_{g_j \in C(xS_\delta y)} w_j^-, \text{ and}$$

$$c^+(x,y) = \sum_{g_j \in C(xS_\delta y)} w_j^+.$$

However, this does not necessarily fulfills constraints (2.3.5) and (2.3.6). To ensure these constraints fulfillment in the definition of $c(x, y)$, it is needed to consider the complete set of criteria, $\mathscr{F}$. By definition, this involves contemplating $C(xS_\delta y)$ and $D(xS_\delta y)$. Thus, considering Constraints (2.3.5) and (2.3.6) in the definition of $c(x, y) = [c^-(x, y), c^+(x, y)]$, we have that

$$c^-(x,y) = \sum_{g_j \in C(xS_\delta y)} w_j^-$$

only if it is true that

$$\sum_{g_j \in C(xS_\delta y)} w_j^- + \sum_{g_j \in D(xS_\delta y)} w_j^- \le 1, \text{ and}$$
$$\sum_{g_j \in C(xS_\delta y)} w_j^- + \sum_{g_j \in D(xS_\delta y)} w_j^+ \ge 1.$$

Otherwise, $c^-(x, y)$ shall be

$$1 - \sum_{g_j \in D(xS_\delta y)} w_j^+.$$

Similarly,

$$c^+(x,y) = \sum_{g_j \in C(xS_\delta y)} w_j^+$$

only if it is true that

$$\sum_{g_j \in C(xS_\delta y)} w_j^+ + \sum_{g_j \in D(xS_\delta y)} w_j^- \le 1, \text{ and}$$
$$\sum_{g_j \in C(xS_\delta y)} w_j^+ + \sum_{g_j \in D(xS_\delta y)} w_j^+ \ge 1.$$

Otherwise, $c^+(x, y)$ shall be

$$1 - \sum_{g_j \in D(xS_\delta y)} w_j^-.$$

Fernandez et al. [97] show that $c^+(x, y) \ge c^-(x, y)$, and that if $C(xS_\delta y) = \emptyset$ then $c(x, y) = [0, 0]$ and if $C(xS_\delta y) = \mathscr{F}$ then $c(x, y) = [1, 1]$.

Let Δ be the set $\left\{\alpha_j \in \mathbb{R} : p(g_j(x) \ge g_j(y)) = \alpha_j;\ j = 1, \cdots, k\right\}$. For each $\delta \in \Delta$ Fernandez et al. [97] state that x outranks y with marginal likelihood index B_δ and majority strength $\lambda = [\lambda^-, \lambda^+]$ ($\lambda^- > 0.5$) if and only if

1. $p(c(x, y) \ge \lambda) \ge \phi$;
2. $1 - \max_{g_j \in D(xS_\delta y)} \{p(g_j(y) \ge g_j(x) + v_j)\} \ge \phi$; and
3. $B_\delta = \max\{\phi\}$ fulfilling 1 and 2.

Where $\delta \ge \phi \in \mathbb{R}$, and $p(\cdot)$ is defined in (2.2.4). With the above notation, it is said that x outranks y with likelihood index $\beta(x, y) \in [0, 1] = \max\{B_\delta\}$ ($\delta \in \Delta$) and majority strength $\lambda = [\lambda^-, \lambda^+]$ ($\lambda^- > 0.5$). If Δ is empty, $\beta(x, y)$ is set to zero. Moreover, it is assumed that the DM uses an implicit likelihood threshold $\beta_0 > 0.5$ such that if $\beta(x, y) \ge \beta_0$ then the assertion "x is at least as good as y" is accepted.

The concept of dominance is also extended in Ref. [97]. In that work, dominance is not crisp, but there is a "degree of credibility", $xD^{\alpha}y$, of the dominance. Let x and y be two solutions and α a real number; y is α-dominated by x, denoted by $xD(\alpha)y$, if and only if $\min_{1 \leq j \leq k} p(g_j(x) \geq g_j(y)) = \alpha \geq 0.5$.

Finally, for each pair $(x, y) \in \mathscr{A} \times \mathscr{A}$, the following preference relations may be defined based on the likelihood index associated with "x is at least as good as y".

- **Strict preference**: $xPy \iff \beta(x, y) \geq \beta_0 \wedge \beta(y, x) < 0.5$,
- **Weak preference**: $xQy \iff \beta(x, y) \geq \beta_0 \wedge 0.5 \leq \beta(y, x) < \beta_0$,
- **K preference**: $xKy \iff \beta(y, x) < 0.5 < \beta(x, y) < \beta_0$,
- **Indifference**: $xIy \iff \beta(x, y) \geq \beta_0 \wedge \beta(y, x) \geq \beta_0$,

where $\wedge$ is the conjunction operator.

2.3.5 Eliciting the decision maker's system of preferences

Due to the common conflicting nature of criteria in multicriteria problems, the total number of mathematically equivalent solutions can be very large or even infinite. However, the DM usually is interested only in some preferred solutions instead of all these solutions. The information about the DM's preferences can be used in order to guide the search towards the *most preferred solution* [138,206]. That is, incorporating the DM's preferences to the search procedure before the actual search makes the optimization method to recommend just the most preferred solution alternatives. Using this method, multiobjective optimization problems (described below) can become scalar objective optimization problems. Then, a method to optimize a scalar objective is applied to find the optimal solution.

The required preferential information can be specified either through direct procedures in which a decision analyst elicits it directly from the DM, or through indirect procedures in which the DM provides examples of holistic decisions. Then, such decisions are evaluated to determine the required preferential parameters that are most consistent with the DM's decisions. Generally, the preferential parameters used by the approaches presented in the literature to model the DM's system of preferences are rather difficult for the DM to provide. This difficulty arises from at least one of four sources: a) the DM's decision policy does not match exactly with the model's assumptions and mathematical structure; b) the DM's decision policy is poorly-defined (e.g. a heterogeneous group); c) the DM is a mythical or inaccessible person; and d) lack of capacity of the DM to express the implicit parameters of his/her decision policy in numerical form. Indirect elicitation methods (which compose the so-called Preference Disaggregation Analysis paradigm) is an alternative. This analysis infers the decision model's parameters from holistic decisions provided by the DM and use regression-like methods to produce a decision model as consistent as possible with the set of reference (training) decisions.

Preference Disaggregation Analysis

Preference Disaggregation Analysis (PDA) methods analyze decisions made by the DM in order to identify the aggregation model that underlies the outcome of the known decisions. Such decisions create a so-called reference set. Usually, this set could be [144]:

- a set of past decisions;
- a subset of decisions on current alternative solutions, especially when there are many of those;
- a set of fictitious actions, consisting of performances on the criteria which can be easily judged by the decision maker to perform global comparisons.

The PDA paradigm is of growing interest because it requires less cognitive effort from the DM. The underlining reason is that DMs often feel more comfortable about making decision judgments than about explaining them. The DM is asked to provide a ranking or a classification of the reference alternatives according to his/her decision policy (global preferences). Then, using regression-based techniques the global preference model is estimated so that the DM's global evaluation is reproduced as consistently as possible by the model.

Indirect elicitation approaches have been used for decades to build functional or utility decision models (e.g. [156,226,268]). In MCDA, Jacquet-Lagreze and Siskos [143] pioneered the UTA method. In the frame of outranking (see Subsection 2.3.4) approaches, using indirect methods is even more important, because the DM must set parameters that are very unfamiliar to her/him (e.g., veto thresholds). In this frame, Richard [238] was one of the first authors trying to infer the parameters of an outranking-based method, ELECTRE III (cf. [144]). Some important references are the works of Mousseau and Slowinski [216], Doumpos et al. [79], and Fernandez et al. [99]. Zopounidis and Doumpos showed in Ref. [308] that the methods using PDA obtained results that were clearly superior on several tests. All these proposals identify punctual values for the model's parameters, which are supposed to be appropriate to explain or suggest new decisions. This is particularly important in problems with four or more objectives, because the capacity of the human mind is limited to handling a small amount of information at the same time [207].

2.4 Multiobjective optimization

2.4.1 Multiobjective optimization problems

A multiobjective (multicriteria) optimization problem can be defined as finding the values for all the decision variables contained in a solution vector[2] in such a way that they satisfy a list of constraints and optimize a function vector of k objective (criteria) functions[3] according to the decision maker's preferences. These objective functions express the criteria used to evaluate how "good" a potential action is and they are often in conflict with each other. In this work, the Portfolio Optimization Problem is a multicriteria optimization problem with $n \geq 2$ decision variables, $k \geq 2$ objective functions, m inequality constraints and p equality constraints. The Portfolio Optimization Problem requires a "good" compromise (from the decision maker's point of view) to be obtained among the k criteria. Such compromise indicates here, without loss of generality, maximization of the impact on the criteria. In what follows $\mathscr{A}$ denotes a set of portfolios. Each portfolio $x \in \mathscr{A}$ is composed of n decision variables:

$$x = [x_1, x_2, \cdots, x_n]^\top.$$

Such decision variables represent the numerical values that characterize the amount of resources allocated by a portfolio and that must be found to solve the optimization problem. These numerical values are known as attributes and are assigned to the decision variables in such a way that a list of constraints (of equality and inequality) is fulfilled:

$$b_i(x) \leq 0; i = 1, 2, \cdots, m;$$

$$d_j(x) = 0; j = 1, 2, \cdots, p.$$

The portfolios that fulfill these constraints are considered feasible and form the feasible space of solutions (Ω).

It is possible to consider three euclidean spaces: the decision space, the objective space and the criteria space. The decision space (solution space) is made up of n dimensions and contains the solution vectors of n decision variables. The objective space contains M dimensions and shows the impact on the M objectives established by the decision maker. Finally, the criteria space is made up of k dimensions and specifies the area on which the portfolio performances from the solution space are plotted. Each axis of the criteria space corresponds to a component of the criteria vector g, composed of k criteria functions:

$$g(x) = [g_1(x), g_2(x), \cdots, g_k(x)]^\top.$$

[2]We will refer to a solution vector as simply *solution*. We will also indistinctly use *potential action* or *portfolio* when addressing the Portfolio Optimization Problem.

[3]We will refer to objective functions indistinctly as criteria functions (or simply as *criteria*).

The objective function is a formal representation of what is to be optimized and depends on the variables to be established within the optimization. An example of an objective function is the risk formulation of the portfolio problem, viz.

$$\underset{x\in\Omega}{\text{maximize}}\, x^\top \zeta - \psi x^\top \Sigma x,$$

subject to

$$x^\top l = 1, \qquad l^\top = [1, 1, \cdots, 1].$$

Where x represents the vector of weights indicating the support given to the investment objects (vector solution, variables to be established), $\zeta = \{\mathbb{E}(r_1), \cdots, \mathbb{E}(r_n)\}^\top$ is the vector of expected returns of the n investment objects; ψ is an indicator of the decision maker's risk aversion, and Σ is the co-variance matrix of the alternatives.

In this case, the only objective function is given by

$$g_1(x) = x^\top \zeta - \psi x^\top \Sigma x.$$

Without loss of generality, we say that a criteria vector $g(x)$ dominates another criteria vector $g(y)$ (denoted as $g(x) \succeq g(y)$) if and only if $g_i(x) \geq g_i(y)$ for all i and exists a j such that $g_j(x) > g_j(y)$; $i, j \in 1, 2, \cdots, k$.

A feasible solution vector x ($x \in \Omega$) is Pareto-efficient if and only if there is no other feasible solution vector y such that $y \succeq x$.

The Pareto front (PF) is the projection of the set of Pareto-efficient solutions in the criteria space. Formally:

$$\text{PF} = \{g(x) \,:\, x \text{ is Pareto-efficient}\}.$$

Usually, finding the PF is not an easy task and, sometimes, it is impossible. Therefore, it is common to use techniques that allow approximations to the PF.

Usually, the Portfolio Optimization Problem is defined with constraints representing relevant characteristics. These constraints are related to the transaction costs, the cardinality of the portfolio, the difference in the portfolio's expected return with respect to the expected return of a standard index, the proportion of the investment destined to specific alternatives or industries (e.g., R&D and social responsibility), the risk implied in the portfolio, the minimum number of transactions, the investment amounts per operation and round portions (e.g., transactional amounts of alternatives in multiples of 50, 100, 200, $\cdots$). The Portfolio Optimization Problem is an instance of the family of quadratic programming problems when Markowitz's standard model of mean-variance is considered. But if this model is generalized to include cardinality and delimitation constraints, then there is no exact algorithm capable of solving the portfolio problem in an efficient manner. Heuristic (and specifically meta-heuristic) algorithms in this case are essential.

2.4.2 Multiobjective evolutionary algorithms

Meta-heuristics are high-level procedures or heuristics designed to find, generate, or select a heuristic (partial search algorithm) that can provide a good enough solution to optimization problems (relatively close to the global optimum). They are especially useful with incomplete or imperfect information or a limited computing capacity [35]. Most meta-heuristics make use of stochastic components (which involve random variables) and do not use the gradient or Hessian matrix of the objective function, so they offer the advantage that the function to be optimized does not need to be continuously differentiable. Furthermore, the Hessian matrix tends to make the optimization process slow as the number of decision variables increases [39]. Meta-heuristics are capable of dealing with complex and large search spaces. They have proven to be an effective tool for solving *hard* optimization problems (problems that can not be solved optimally in acceptable time periods); they provide a balance between “good” solutions and an affordable computational time and cost. However, meta-heuristics are usually based on specific characteristics of the problem in question, which makes its design and development a complex task. This is because the number of parameters of the meta-heuristics has a direct effect on the complexity of the algorithm, which complicates the analysis. Moreover, the efficiency of the meta-heuristics depends on the operators provided by the user, and the best alternatives for these operators can only be formalized by experts in the domain of the problem and the meta-heuristics used.

Compared to optimization methods based on mathematical programming, meta-heuristics offer some relevant advantages, from which the following stand out:

- Meta-heuristics are less sensitive to the mathematical properties of the objective function and the constraints of the problem [66].
- Some meta-heuristics allow to approximate the Pareto front in a single run of the algorithm.

From these tools, the so-called evolutionary algorithms have proven to be one of the most efficient and reliable methods due to the wide range of advantages they provide with respect to other optimization tools. Moreover, evolutionary algorithms have a satisfactory behavior when dealing with problems with large dimensions (regarding the number of alternatives and criteria).

Multiobjective evolutionary algorithms (MOEAs) are used to solve problems characterized by having multiple criteria, problems with considerably large search spaces and whose solutions require risk management or uncertainty. The MOEAs are characterized by working with several solutions at the same time, where each of these solutions satisfies to some extent each of the criteria. This allows them to work trying to satisfy several criteria at the same time.

MOEAs belong to a class of meta-heuristic optimization algorithms based on a population; they usually use mechanisms inspired by biological evolution: reproduction, mutation, recombination, natural selection and survival of the fittest, to solve optimization problems. The candidate solutions to the optimization problem play the role of individuals in a population, and the objective function determines the environment within which the solutions "live". Normally, MOEAs include four types of algorithms: genetic algorithms, genetic programming, evolutionary programming and evolutionary strategy. Out of the four types, genetic algorithms (GA) are the most popular. In genetic algorithms, the solution to a problem is sought in the form of strings of characters (the best representations are usually those that reflect something about the problem that is addressed), virtually always applying recombination operators such as crossing, selection and mutation operators. GA is one of the most popular meta-heuristics applied to the Portfolio Optimization Problem.

Although genetic algorithms represent viable ways to solve models related to the selection of portfolios, some problems arise during their implementation. These problems are related to the inherent nature of the Portfolio Optimization Problem, mainly the imposition of multiple constraints. The main problem of applying genetic algorithms to optimization problems with constraints is how to deal with these constraints. This can be done with strategies such as rejecting, repairing or penalizing solutions that do not fulfill the constraints. An even more relevant disadvantage of genetic algorithms is the selection of parameters. Because these algorithms require various parameters for which advanced knowledge of the particular problem may be required (degree of mutation, crossing points, representation of the solutions), this approach is often criticized when compared with other methods less dependent on the input parameters.

Following Ref. [253], genetic algorithms encode candidate solutions in chains of characters of finite length and of certain cardinality called chromosomes; the characters are called genes and the values of these allele. A gene corresponds to a variable x_i, and a chromosome corresponds to a solution represented in a set of genes $x = (x_1, x_2, \cdots, x_n)$ if there are n decision variables. In the problem of non-linear 0-1 programming, a gene can be easily represented by two generic binary codes, 0 or 1. In the Portfolio Optimization Problem, $x_i = 0$ means that the ith investment object is not supported by the portfolio, $x_i = 1$ means that the object will be supported with all the available resources and any value from $(0, 1)$ represents that the object will receive a part of the resources.

To achieve good solutions and implement natural selection through evolution, it is necessary to implement a measure to distinguish the quality between solutions. This measure could be an objective function where the best solutions are chosen over the worst ones. The measure of quality, or fitness, must determine the relative aptitude of a solution that will subsequently be used in the genetic algorithm to guide the evolution towards good solutions.

The size of the population, which is regularly specified by the user, is an important factor that affects the scalability and performance of genetic algorithms. For example, small sizes of the population can lead to premature convergence and poor performance of the solutions. On the other hand, large sizes of the population may lead to unnecessary spending computational time.

Once the problem has been encoded in chromosomes and the measure of aptitude to discriminate the good solutions from the bad ones has been chosen, it is possible to start with the evolution of solutions to the problem using the following steps (see [292]):

- Produce the initial population. The initial population of candidate solutions is usually generated randomly throughout the search space. However, domain-specific knowledge or any other information can be easily incorporated.
- Apply the crossing operation to the initial chromosomes. The main objective of the crossing operation is to generate a diverse offspring of chromosomes to obtain a better solution than their parents. The offspring created by the crossing operation will not be identical to any particular parent, and instead parental features will be combined in a new way [122]. There are several ways to achieve this, and a competent performance depends on a correctly designed crossover mechanism, however there are some common techniques to create the offspring of chromosomes: one point, two points and multiple points. The "one point" crossing technique randomly selects a crossing point within the chromosome, then the parent chromosomes are exchanged at this point to produce new offspring chromosomes.
- Perform the mutation operation on the chromosomes. While the crossing operation starts with two or more parent solutions to create a new one, the mutation changes one solution randomly but locally. A mutation operation is a method to create a new chromosome from another chromosome in the population. The main objective of the mutation operation is to prevent the genetic algorithm from converging too quickly in a small area of the search space. Usually, the mutation operation occurs with a given probability and at a random point on the chromosomes. In the case of binary chromosomes, the new chromosome can be generated based on randomly changing a gene from '0' to '1' or vice versa. In the case of real-valued alleles, a random generation of such value can be assigned to an also randomly chosen gene.
- Use the selection operation to create the population with the greatest aptitude. After performing the crossing and mutation operations (operations that create a new population), the next operation will be to select the chromosomes with (perhaps) the highest fitness value. The selection operation assigns more probability of being selected to those solutions with higher fitness values imposing a survival mechanism of the fittest within the candidate solutions. There are several selection procedures although such selection is usually made through the roulette approach, in which the chromosomes are assigned to the search space in a roulette wheel proportional to their fitness. The goal of this is that the fittest chromosome is more likely to be selected although elitism may not be ensured.

- Replace the original solutions. The population created by selection, crossing, and mutation replaces the original parent population. Several replacement techniques such as elitist replacement, intelligent generation replacement, and steady-state replacement methods are used in genetic algorithms.

Multiobjective evolutionary algorithm based on decomposition

With the aim of aiding in the selection process, MOEAs find a subset of the Pareto-efficient solutions that is provided to the DM who is in charge of selecting the best alternative according to his/her own preferences. In order to present a representative subset of alternatives, MOEAs look for a manageable number of Pareto-efficient solutions which are evenly distributed along the Pareto front (PF), and thus are good representatives of the entire PF. The fitness measure in MOEAs based on the Pareto dominance concept is determined by the individual's Pareto dominance relations with respect to other individuals. Using this fitness measure alone discourages the diversity of the search [175]. Several other efforts have been made in order to discover complementary fitness measures.

One of the lines in this context is the aggregation of criteria. The idea is that a solution to the original problem could be an optimal solution of a single criterion optimization problem in which the criterion is an aggregation function of all the original criteria. Therefore, the approximation to the PF can be decomposed into a number of single objective optimization sub-problems. MOEA/D [295] is a MOEA that implements this idea. The objective in each of the sub-problems that MOEA/D optimizes is an aggregation of all the criteria using different weights for the criteria. Neighborhood relations among these sub-problems are defined based on the distances between their aggregation coefficient vectors. Each sub-problem (i.e., scalar aggregation function) is optimized in MOEA/D by using information mainly from its neighboring sub-problems.

MOEA/D requires a decomposition technique for converting the approximation of the PF of Problem (1.2.1) into a number of single objective optimization problems. In principle, any decomposition approach can serve for this purpose. A common approach used in the MOEA/D context is the Tchebycheff method [205]. A single objective optimization sub-problem in this approach is

$$\underset{x\in\Omega}{\text{minimize}}(f(x|\lambda^j, z^*) = \max_{1\le i\le k}\{\lambda_i^j|g_i(x) - z_i^*|\}) \tag{2.4.7}$$

where $\lambda^j = (\lambda_1^j, \cdots, \lambda_k^j)^\top$, $\lambda_i^j \in \mathbb{R}$, is a weight vector of the criteria that satisfies $\lambda_i^j \ge 0$ for all $i = 1, \cdots, k$ and $\sum_{i=1}^k \lambda_i^j = 1$. $z^* = (z_1^*, \cdots, z_k^*)^\top$ is the reference point; that is, $z_i^* = \max\{g_i(x)|x \in \Omega\}$, for each $i = 1, \cdots, k$. And $g_i(x)$ is the impact in the i-th underlying criterion, as specified in Problem (1.2.1).

Following [175], it is well known that, under mild conditions, for each Pareto optimal portfolio there exists a weight vector such that it is the optimal solution of (2.4.7) and each optimal solution of (2.4.7) is a Pareto optimal solution of problem (1.2.1).

If $\lambda^1, \cdots, \lambda^N$ is a set of weight vectors, then we have N single objective optimization sub-problems. If N is reasonably large and $\lambda^1, \cdots, \lambda^N$ are properly selected, then the optimal solutions to these sub-problems will provide a good approximation to the PF of Problem (1.2.1) [295]. Algorithm 1 shows a simple algorithm of MOEA/D.

Algorithm 1 Multiobjective evolutionary algorithm based on decomposition, MOEA/D

Require: N, the number of objective optimization sub-problems; and T, the number of closest weight vectors to λ^i.

Ensure: A final population, $x^1, \cdots, x^N$.

1: For each $i = 1, \cdots, N$, set the indexes of the T closest weight vectors to λ^i (computed through the Euclidean distance) as $B(i) = i_1, \cdots, i_T$; where $\lambda^{i_1}, \cdots, \lambda^{i_T}$ is known as the neighborhood of λ^i.

2: Generate an initial population $x^1, \cdots, x^N$.

3: Initialize $z^* = (z_1^*, \cdots, z_k^*)^\top$, or a corresponding approximation.

4: **for** $i = 1, \cdots, N$ **do**

5: Randomly select two indexes k, l from $B(i)$, then generate a new solution $\hat{y}$ from x^k and x^l by using genetic operators, and apply a mutation operator to $\hat{y}$.

6: Apply a problem-specific repair/improvement heuristic on $\hat{y}$ to produce y.

7: For each $j = 1, \cdots, k$, if $z_j^* < g_j(y)$, then set $z_j^* = g_j(y)$.

8: For each index $j \in B(i)$, if $f(y|\lambda^j, z^*) < f(x^j|\lambda^j, z^*)$, then set $x^j = y$.

9: **end for**

10: If the stopping criterion is satisfied, then stop and output $x^1, \cdots, x^N$. Otherwise, go to Step 4.

Preference-based multiobjective evolutionary algorithms

Evolutionary multi-criteria optimization algorithms (whose performance analyzing data have been validated in different fields, e.g., in Refs. [230,231]) work with a population of solutions and can approximate a set of trade-off alternatives simultaneously. They have been widely accepted as a major tool for addressing the problem of finding “good” portfolios. The main goal of this type of algorithms is finding a set of efficient solutions that approximate the true Pareto front in terms of convergence and diversity. The intervention of the DM is thus not traditionally used in the process. So, rather little interest has been paid in the literature to choosing one of the efficient solutions as the final one in contrast to the interest paid in approximating the whole Pareto front.

Recently, however, the interest in incorporating the DM's preferences during the multi-objective optimization process has increased. Fonseca and Fleming [107] probably suggested the earliest attempt to incorporate preferences; their proposal was to use MOGA together with goal information as an additional criterion to assign ranks to the members of a population. More recently, the idea of measuring the preference-based distance of the potential solutions with respect to a reference point has gained much interest (see e.g., Refs. [176,211,284]). Nevertheless, some of them are ad-hoc methodologies and/or treat points outside the preferred region as equally redundant. Particularly, the so-called R-metric [176] is a very recent and interesting idea.

Three classes of multi-objective optimization methods can be identified according to the role of the DM in the solution process when he/she is available (cf. Refs. [141,205]). In a *priori methods*, the DM articulates her/his preference information and hopes before the solution process. The difficulty here is that the DM does not necessarily know the limitations and possibilities of the problem and may have too optimistic or pessimistic hopes. Alternatively, a set of Pareto optimal solutions can be generated first and then the DM is supposed to select the most preferred one among them. Typically, evolutionary multi-objective optimization algorithms do this *a posteriori* way. However, it may not be suitable for the Portfolio Optimization Problem addressed in this work given that if there are more than three criteria defined as interval numbers in the problem, it may be difficult for the DM to analyze a large amount of information. On the other hand, generating the set of efficient solutions may be computationally expensive. Furthermore, supplying the DM with a large amount of trade-off points provides many irrelevant or even noisy information to the decision-making procedure. Another alternative is that, after each iteration, the DM is provided with one or more efficient solutions that obey the preferences expressed as well as possible and he/she can specify his/her preference information on them in such a way that this information is considered for the next iteration. This seems to be the ideal way to incorporate the DM's preferences into the search process. However, there might be situations where the DM is not willing/capable to get involved in the procedure. Hence, one of the other two ways to incorporate the preferences must be implemented.

Besides using different methods to provide preference information, multi-objective optimization algorithms also differ from each other in the type of information that is utilized in generating new, improved solutions and what is assumed about the behavior of the DM. Perhaps the most intuitive one is the weighting method, which assigns a relative importance to each criterion: the larger the weight is, the more important the criterion is. Zitzler et al. [304] used this method combined with the hyper-volume indicator [305] in order to guide the search based on the DM's preferences expressed by weighting coefficients or a reference point. Deb [70] developed a modified fitness-sharing mechanism, by using a weighted Euclidean distance, to bias the population distribution. In Ref. [49], Branke and Deb modified the crowding distance calculation in NSGA-II by using a weighted mapping method in order to focus the search on the preferred part of the Pareto front. Another method to use the DM's preferences modifies the original Pareto dominance by classifying criteria into different levels and priorities (e.g., [50,106,148]); thus, creating a ranking of the criteria. A convenient way to perform such ranking is by providing the DM with pairs of criteria and asking her/him to provide a decision about which one is the most important. A relevant limitation with this method is that there might be situations where incomparability exists and creating the whole ranking may become a difficult problem to solve. The third approach combines the classical reference point-based method [285] with evolutionary multi-objective optimization (e.g., [71,176,236]). In such methodology, the DM supplies for each criterion the level that should be achieved according to her/his preferences. The reservation level corresponds to the worst value for which the DM is still satisfied [28]. This method is the most used in the related literature [28]. Another methodology to incorporate the DM's preferences is exploiting the outranking concept [244], which states the credibility index of the statement "solution x is at least as good as solution y" (see e.g., Refs. [95,98]). This is a very convenient way to incorporate the DM's preferences since she/he usually considers more information than just the relative importance of the criteria in order to make decisions. The outranking methods can handle intransitive preferences, incomparability, veto effects, and even qualitative and ordinal information for some criteria. Furthermore, through an adequate way of preference parameters elicitation, the DM's decision policy can be reproduced during the optimization procedure in such a way that the most preferred solution can be found and an arduous work by the DM can be avoided.

2.5 Fuzzy Logic

The usefulness of the conventional dual logic for modeling purposes is undisputed. However, there are limits to the possibility of using this logic. Particularly, because real situations are often uncertain or vague concerning the description of the semantic meaning of the events, phenomena, or statements. Vagueness is, together with uncertainty, the modeling goal of Fuzzy Logic [83].

Often, Fuzzy Logic is used in the context of the Portfolio Optimization Problem when it is difficult to use Probability Theory. Fuzzy Logic is commonly used with the aim of reflecting the vagueness (or ambiguity) in the definition of returns. One of the pioneering works of the Fuzzy Logic in the context of the Portfolio Optimization Problem was the one made by Watada in Ref. [283]. He extended Markowitz's media-variance approach to the fuzzy environment. Huang proposed in Ref. [140] two models based on fuzzy semi-variance as a risk measure using a genetic algorithm based on fuzzy simulation to provide a general solution to these new problems. Wang et al. [282] use the concept of Fuzzy Value at Risk as a risk measure in a portfolio optimization model where portfolio returns follow imprecise distributions. They use a particle optimization algorithm to approximate optimal solutions. Huang [139] proposed a model based on fuzzy entropy as the risk measure to be minimized and used a genetic algorithm to look for solutions close to the optimal ones. Man Hui et al. [191] incorporate the fuzzy concept in the portfolio problem in order to include expert knowledge in the form of inaccurate information.

Fuzzy Logic [293] mainly focuses on modeling problems where classical approaches as Set Theory and Probability Theory are insufficient or non-operational. It generalizes the classical notion of set and proposition by introducing the concept of fuzziness through the so-called fuzzy set concept. Zadeh [293] states that "the notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables". Some of the main goals of Fuzzy Logic are [303]:

- **Modeling of uncertainty**. The uncertainty modeled by Fuzzy Logic is in the sense of vagueness rather than the lack of knowledge about the value of a parameter (as in tolerance analysis).
- **Relaxation** Classical models and methods are normally based on dual logic. Often this view does not capture reality adequately. Fuzzy Logic has been used extensively to relax or generalize classical methods to a gradual character.
- **Compactification**. Due to the limited capacity of the human short term memory or of technical systems it is often not possible to either store all relevant data, or to present masses of data to a human observer in such a way that he or she can perceive the information contained in these data. Fuzzy Logic has been used to reduce the complexity of data to an acceptably degree usually either via linguistic variables or via fuzzy data analysis (fuzzy clustering, etc.).
- **Meaning Preserving Reasoning**. Expert system technology has already been used since two decades and has led in many cases to disappointment. One of the reasons for this might be that expert systems in their inference engines, when they are based on dual logic, perform symbol processing (truth values true or false) rather than knowledge processing. In approximate reasoning meanings are attached to words and sentences via linguistic variables. Inference engines then have to be able to process meaningful linguistic expressions, rather than symbols, and arrive at membership functions of fuzzy sets, which can then be re-translated into words and sentences via linguistic approximation.

2.5.1 Basic definitions

A classical (crisp) set is normally defined as a collection of elements or objects $x \in X$. Each single element can either belong to or not belong to a set A, $A \subseteq X$. In the former case, the statement "x belongs to A" is true, whereas in the latter case this statement is false. In the characteristic function describing such classical set, 1 indicates membership and 0 non-membership. That is:

$$A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

For a fuzzy set, the characteristic function allows various degrees of membership for the elements of a given set. If X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X, \mu_{\tilde{A}}(x) \in [0, 1]\}.$$

That is, $\tilde{A}$ on X is characterized by a membership function that transforms the elements of a domain, space or universe of discourse X in the interval [0, 1], $\mu_{\tilde{A}} : X \longrightarrow [0, 1]$. Thus, generalizing the concept of classical set whose membership function takes values from {0,1}.

The support of a fuzzy set $\tilde{A}$, $Support(\tilde{A})$, is the (crisp) set of all $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$. The (crisp) set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the α-level set:

$$A_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}.$$

For a finite fuzzy set $\tilde{A}$, the cardinality $|\tilde{A}|$ is defined as

$$|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x).$$

$||\tilde{A}|| = \frac{|\tilde{A}|}{|X|}$ is called the relative cardinality of $\tilde{A}$.

2.5.2 Fuzzy Linguistic Approach

Fuzzy Linguistic Approach is an approximate approach that has a theoretical basis on Fuzzy Logic. It represents qualitative aspects as linguistic values through linguistic variables [294]. A linguistic variable is a variable whose values are words or sentences in a natural or artificial language. Such concept is particularly important in situations where information may not be easily assessed quantitatively but it can be easily assessed qualitatively. A classical example is the evaluation of subjective perception through natural language: too much, high, pretty, fast, etc. The convenience of using qualitative labels instead of numerical values is provided by two aspects [57]. First, there are situations in which the information can be non-quantifiable due to its nature, and thus, can only be "measured" using linguistic terms. But there are other occasions in which quantitative information can not be measured because the necessary elements are not available to carry out an exact measurement of that information, or because the cost of its measurement is very high. Therefore, using an "approximate value" is accepted.

Following Ref. [294], a linguistic variable is characterized by a quintuple (H, $T(H)$, U, G, M) in which H is the name of the variable; $T(H)$ is the *term-set* of H, that is, the collection of its linguistic values; U is a universe of discourse; G is a syntactic rule which generates the terms in $T(H)$; and M is a semantic rule which associates with each linguistic value X its meaning, $M(X)$, where $M(X)$ denotes a fuzzy subset of U. The meaning of a linguistic value X is characterized by a compatibility function, $c : U \longrightarrow [0, 1]$, which associates with each u in U its compatibility with X. Thus, the compatibility of age 27 with *young* might be 0.7, while that of 35 might be 0.2.

There are two possibilities to choose the appropriate linguistic descriptors of the term-sets and their semantics. The first possibility is to define the set of linguistic terms by a context-free grammar, and its semantics by fuzzy numbers described by a parameterized membership function and semantic rules [44,294]. The second possibility defines the set of linguistic terms using an ordered structure of labels, and the semantics of the linguistic terms is derived from the ordered structure itself, which can be uniformly distributed in the interval [0, 1] or not [45].

2.5.3 Set-Theoretic Operations for Fuzzy Sets

As stated above, Fuzzy Logic is a generalization of the classical dual logic. Thus, the evident question about the behaviour of the former with respect to the classical Set Theory operations arises. Here, we present and describe some operators used to extent the classical Set Theory in order to deal with fuzzy sets.

As in the classical Set Theory, the main operations with fuzzy sets are *intersection* ($\cap$), *union* ($\cup$), and *complement* ($\complement$). In the classical Set Theory such operations are defined for two (crisp) sets, A and B, as

$$A \cap B = \{a | a \in A \text{ and } a \in B\}$$

$$A \cup B = \{a | a \in A \text{ or } a \in B\}$$

$$A^{\complement} = \{a | a \notin A\}$$

Let $v(P)$ be the evaluation of proposition P with values in $\{0, 1\}$. The above operations can be represented as:

$$v(a \in A \cap B) = v(a \in A) \wedge v(a \in B),$$

$$v(a \in A \cup B) = v(a \in A) \vee v(a \in B),$$

$$v(a \in A^{\complement}) = \neg v(a \in A).$$

Where $\wedge$, $\vee$ and $\neg$ represent conjunction, disjunction and negation, respectively.

Now, if for each element a, the membership degree to set A (i.e., $a \in A$) is a number of $[0,1]$ instead of $\{0,1\}$, then the operators used to represent conjunction and disjunction are no longer satisfactory; new operators extending the previous ones are needed. This extension requires taking into account that the logical connectors valued in $\{0,1\}$ must be special cases of the ones valued in $[0,1]$. Several operators have been proposed with this purpose, where membership function plays a (naturally) crucial role. Let us now present some of the most outstanding ones.

Traditional operators

We shall first present the concepts suggested by Zadeh in Ref. [293]. They constitute a consistent framework for the theory of fuzzy sets. They are, however, not the only possible way to extend classical set theory consistently. Zadeh and other authors have suggested alternative or additional definitions for set-theoretic operations. The operators suggested by Zadeh in Ref. [293] are:

- Complement. The membership function of the complement of a fuzzy set $\tilde{A}$, is defined by
$$\mu_{\tilde{A}^{\complement}}(x) = 1 - \mu_{\tilde{A}}(x).$$
- Intersection. The membership function of the intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined by
$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}.$$
- Union. The membership function of the union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined by
$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}.$$

The suggestions to use other operators besides the traditional ones vary with respect to the generality or adaptability of the operators as well as to the degree to which and how they are justified. Justification ranges from intuitive argumentation to empirical or axiomatic justification [303]. The most well-known alternative operators are the so-called triangular norms and co-norms for conjunction and disjunction, respectively.

Triangular norms and co-norms

Triangular norms (*t-norms*) are two-valued functions from $[0,1] \times [0,1]$ into $[0,1]$ that define a general class of intersection operators for fuzzy sets and satisfy the following conditions:

1. $v(P_1 \wedge P_2)$ depends only on the values $v(P_1)$ and $v(P_2)$.
2. If $v(P_1) = 1$ then $v(P_1 \wedge P_2) = v(P_2)$ for any P_2.
3. $v(P_1 \wedge P_2) = v(P_2 \wedge P_1)$.
4. If $v(P_1) \leq v(P_2)$ then $v(P_1 \wedge P_3) \leq v(P_2 \wedge P_3)$ for any P_3.
5. $v(P_1 \wedge (P_2 \wedge P_3)) = v((P_1 \wedge P_2) \wedge P_3)$.

Where P_1 and P_2 are propositions. Analogously, a general class of aggregation operators for the union of fuzzy sets called triangular co-norms or t-co-norms (sometimes referred to as s-norms) is defined as follows.

t-co-norms or s-norms are associative, commutative, and monotonic two-placed functions S that map from $[0, 1] \times [0, 1]$ into $[0, 1]$ [303]. These properties are formulated with the following conditions:

1. $v(P_1 \vee P_2)$ depends only on the values $v(P_1)$ and $v(P_2)$.
2. If $v(P_1) = 0$ then $v(P_1 \vee P_2) = v(P_2)$ for any P_2.
3. $v(P_1 \vee P_2) = v(P_2 \vee P_1)$.
4. If $v(P_1) \leq v(P_2)$ then $v(P_1 \vee P_3) \leq v(P_2 \vee P_3)$ for any P_3.
5. $v(P_1 \vee (P_2 \vee P_3)) = v((P_1 \vee P_2) \vee P_3)$.

Where, again, P_1 and P_2 are propositions.

It is easy to see that s is an s-norm if and only if $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \neg s(\neg\mu_{\tilde{A}}(x), \neg\mu_{\tilde{B}}(x))$ is a t-norm. According to this association and using $\neg$ to denote standard negation $\neg\mu_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)$, the following examples of t-norms and s-norms can be given (see Ref. [303]):

$$t_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \begin{cases} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} & \text{if } \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$s_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \begin{cases} \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} & \text{if } \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

$$t_1(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \max\{0, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 1\}.$$

$$s_1(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \min\{1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)\}.$$

$$t_{1.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}{2 - [\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)]}.$$

$$s_{1.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)}{1 + \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}.$$

$$t_2(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x).$$

$$s_2(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x).$$

$$t_{2.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}.$$

$$s_{2.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 2\mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}{1 - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}.$$

$$t_3(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}.$$

$$s_3(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}.$$

These operators are ordered as follows:

$$t_w \leq t_1 \leq t_{l.5} \leq t_2 \leq t_{2.5} \leq t_3, \text{ and}$$
$$s_3 \leq s_{2.5} \leq s_2 \leq s_{1.5} \leq s_1 \leq s_w.$$

Thus, t-norms defined with the previous operators satisfy property (2.5.8) and s-norms satisfy property (2.5.9):

$$t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}. \tag{2.5.8}$$

$$s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \geq \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}. \tag{2.5.9}$$

It may be desirable to extend the range of the previously described operators in order to adapt them to contexts where it is required. This is one of the main aspects of the so-called Compensatory Fuzzy Logic.

2.5.4 Compensatory Fuzzy Logic

The minimum operator is not sensitive to changes in the truth values of the component predicates, which is a limitation to solve problems of classification and selection [86]. Other t-norms do not have this difficulty, but t-norms that lead the conjunction to values lower than the minimum reflect behaviors that are too pessimistic. Operators with which low truth values of a component predicate could be compensated with high truth values of others, would reflect a more optimistic behavior. Espin et al. [86] proved that this permissive approach to compensation, seen as a decision making tool, is better in line with the multi-criteria value theory. Picos [229] demonstrates that real decision makers behave compensatory in certain situations (although, in others, the conjunction is less than the minimum). These ideas are the inspiration of the so-called Compensatory Fuzzy Logic. Let us now briefly describe the axioms that describe the Compensatory Fuzzy Logic (see [83,86]).

Let $a = (a_1, a_2, \cdots, a_n)$, $b = (b_1, b_2, \cdots, b_n)$, $z = (z_1, z_2, \cdots, z_n)$ be any three elements of the Cartesian product $[0, 1]^n$. The quartet of operators (c, d, o, n), where $c : [0, 1]^n \longrightarrow [0, 1]$ is the conjunction operator, $d : [0, 1]^n \longrightarrow [0, 1]$ is the disjunction operator, $o : [0, 1]^n \longrightarrow [0, 1]$ is the order operator and $n : [0, 1] \longrightarrow [0, 1]$ is the negation operator, constitute a Compensatory Fuzzy Logic if the following group of axioms is satisfied:

1. *Compensation axiom*

   $\min(a_1, a_2, \cdots, a_n) \le c(a_1, a_2, \cdots, a_n) \le \max(a_1, a_2, \cdots, a_n)$, and

   $\min(a_1, a_2, \cdots, a_n) \le d(a_1, a_2, \cdots, a_n) \le \max(a_1, a_2, \cdots, a_n)$.

2. *Symmetry (commutativity) axiom*

   $c(a_1, a_2, \cdots, a_i, \cdots, a_j, \cdots, a_n) = c(a_1, a_2, \cdots, a_j, \cdots, a_i, \cdots, a_n)$, and

   $d(a_1, a_2, \cdots, a_i, \cdots, a_j, \cdots, a_n) = d(a_1, a_2, \cdots, a_j, \cdots, a_i, \cdots, a_n)$.

3. *Strict growth axiom*

   If $a_1 = b_1, \cdots, a_{i-1} = b_{i-1}, a_{i+1} = b_{i+1}, \cdots, a_n = b_n$ are greater than zero and $a_i > b_i$ then

   $c(a_1, a_2, \cdots, a_i, \cdots, a_n) > c(a_1, a_2, \cdots, a_i, \cdots, a_n)$, and

   $d(a_1, a_2, \cdots, a_i, \cdots, a_n) > d(a_1, a_2, \cdots, a_i, \cdots, a_n)$.

4. *Veto axiom*

   If $a_i = 0$ for any i then $c(a_1, a_2, \cdots, a_i, \cdots, a_n) = 0$, and

   If $a_i = 1$ for any i then $d(a_1, a_2, \cdots, a_i, \cdots, a_n) = 1$.

5. *Fuzzy reciprocity axiom*

   $o(a, b) = n[o, (b, a)]$.

6. *Fuzzy transitivity axiom*

   If $o(a, b) \ge 0.5$ and $o(b, z) \ge 0.5$, then $o(a, z) \ge \max(o(a, b), o(b, z))$.

7. *Morgan Laws*

   $n(c(a_1, a_2, \cdots, a_n)) = d(n(a_1), n(a_2), \cdots, n(a_n))$, and

   $n(c(a_1, a_2, \cdots, a_n)) = d(n(a_1), n(a_2), \cdots, n(a_n))$.

8. *Idempotency axiom*

   $c(a, a, \cdots, a) = a$, and

   $d(a, a, \cdots, a) = a$.

The Compensatory Fuzzy Logic operators for conjunction have as limit the minimum operator [300]. Some compensatory logic operators are the arithmetic mean and the geometric mean. The latter is considered as the simplest among the quasi-arithmetic means (cf. [83,86]). Unlike the minimum operator, the geometric mean satisfies the strict growth axiom of the Compensatory Fuzzy Logic [82]. Espin et al. define in Refs. [83,86] a Geometric Mean-based Compensatory Fuzzy Logic that fulfills axioms 1-8 as the following quartet of operators (c, d, o, n):

- $c(a_1, a_2, \cdots, a_n) = (a_1, a_2, \cdots, a_n)^{1/n}$,
- $d(a_1, a_2, \cdots, a_n) = 1 - ((1 - a_1)(1 - a_2) \cdots (1 - a_n))^{1/n}$,
- $o(a, b) = 0.5[c(a) - c(b)] + 0.5$, and
- $n(a) = 1 - a$.

Espin et al. [84] proved that there is an interesting correspondence between the archimedean and compensatory operators. For each arquimedean t-norm there exists a compensatory operator such that $x\widehat{AND}y \geq z\widehat{AND}w$ (where the $\widehat{AND}$ is an archimedean) implies that $x\widetilde{AND}y \geq z\widetilde{AND}w$ where $\widetilde{AND}$ is a compensatory one. In particular, they proved that the geometric mean corresponds in this sense with the product operator.

Chapter 3

Uncertainty management through confidence intervals in portfolio optimization

3.1 Introduction

This chapter describes our proposal of using probabilistic confidence intervals as criteria underlying risky objectives to characterize portfolios. Such characterization is important since it allows the investor to consider not only the forecasted impact of the portfolios but also the risk of not obtaining that impact. Furthermore, our proposal identifies the behavior of the investor when facing risk and gives her/him support depending on her/his own preferences, a crucial aspect when addressing the Portfolio Optimization Problem.

In order to evaluate this proposal, an illustrative application in stock portfolio selection is included. We use as our dataset 13 years of historical monthly prices of stocks in the Dow Jones Industrial Average index (DJIA), including those of the 2008 crisis. Besides, we carry out an extensive evaluation comparing the performance of our proposal with respect to the DJIA index, the Markowitz's mean-variance approach, and other more recent approaches. The results show that our proposal outperforms the other ones and allow us to conclude that, within the context of our experiments, i) our proposal was effective in the allocation of resources in most of the periods considered (156 scenarios), ii) our proposal is appropriate to find portfolios by explicitly considering the investor's attitude facing risk, and iii) the confidence intervals implied a robust measure of risk even for the 2008 crisis.

(a) Representation of an interval with a high value of γ_j; information required by a highly risk-averse investor

(b) Representation of an interval with a small value of γ_j; information required by a lowly risk-averse investor

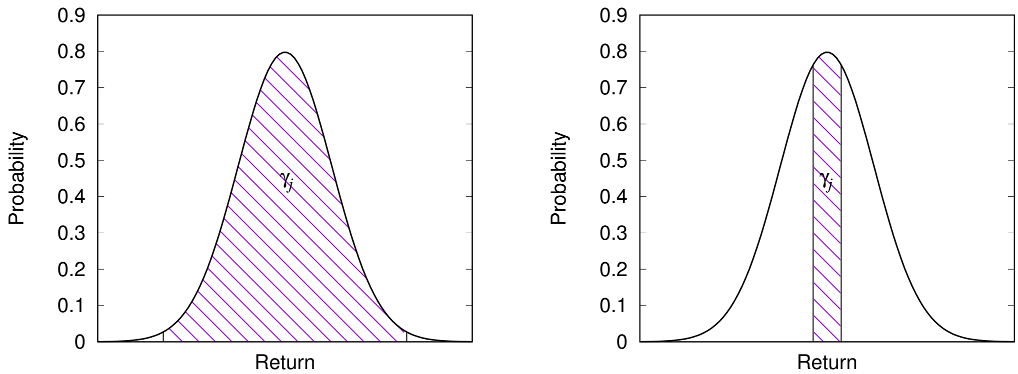


Figure 3.1: Confidence intervals representing types of information required by different behaviours facing risk

The first section of this chapter describes our idea to characterize portfolios when dealing with risky objectives and formalizes the optimization problem to address the idea. Section 3.3 presents a system for the optimization of portfolios where the Multiobjective Evolutionary Algorithm based on Decomposition (MOEA/D) is enhanced to deal with confidence intervals characterizing the portfolios. Finally, Section 3.4 provides an extensive validation of the proposal.

3.2 Problem formalization

Let $R(x)$ be a random variable that represents the return of portfolio x and $\mathbb{P}(\omega)$ the probability that event ω will occur. Then, $\theta_\gamma(x) = [\alpha, \beta] \ : \ \mathbb{P}(\alpha \leq R(x) \leq \beta) = \gamma \ (\alpha, \beta, \gamma \in \mathbb{R})$ is called confidence interval of the portfolio return. We assume that it is possible to consider multiple confidence intervals $\theta_{\gamma_j(x)}$. Furthermore, each value γ_j is selected by the investor according to his/her own preferences. This allows us to incorporate his/her attitude facing risk in the following manner. First, suppose a highly risk-averse investor; such investor would feel more satisfied of making a decision based on intervals with a high probability of containing the actual return. That is, he/she considers more valuable the information about the worst scenarios that could happen when selecting a portfolio; thus, he/she would select high values for γ_j looking for protection against those scenarios (see Figure 3.1a). On the other hand, if the investor is lowly risk-averse, he/she would prefer to make a decision based on intervals that tend to the expected return (see Figure 3.1b).

Therefore, our proposal to characterize the portfolios consists in using confidence intervals as criteria underlying risky objectives. Thus, when the objective is maximization of return, we propose to select the feasible portfolio that maximizes a set of confidence intervals of the portfolio return:

$$\underset{x\in\Omega}{\text{maximize}}(\theta(x) = \theta_{\gamma_1}(x), \theta_{\gamma_2}(x), \cdots, \theta_{\gamma_k}(x)) \tag{3.2.1}$$

where $\theta_{\gamma_j}(x) = \{[\alpha_j, \beta_j] : \mathbb{P}(\alpha_j \leq R(x) \leq \beta_j) = \gamma_j\}$, each value γ_j is specified by the investor and Ω is the set of feasible portfolios. $\mathbb{P}(\omega)$ is the probability that event ω will occur and can be approximated through the frequentist approach.

It is important to note that the maximization referred to in Problem (3.2.1) is not necessarily related to the wideness of the intervals, but it is based on the possibility function defined in Equation (2.2.4). That is, portfolios with the rightmost confidence intervals are preferred.

Each confidence interval $\theta_{\gamma_j}(x)$ is easily understandable even for an investor (decision maker, DM) without a sophisticated technical preparation, since it represents the probability that the return of portfolio x actually lies within the interval $[\alpha_j, \beta_j]$. This is not the case if one considers the technical criteria used in the mean-variance approach [195] or higher statistical moments [76,130,252,256].

Moreover, by using our proposal, the investor can define as many criteria per objective as he/she wishes; thus, the information describing the distribution is enough to satisfy his/her requirements. Nevertheless, we believe that no more than one or two criteria are sufficient to satisfy his/her requirements for information. This is because of the definition of each $\theta_{\gamma_j}(x)$, which allows the proposed approach to encompass multiple points of the probability distribution in a single criterion. That is, in a single criterion we know with a given probability that the portfolio's return can be any of the values within the corresponding interval. This is not possible in point estimators, where the statistical information relies on just one point. In some approaches (see e.g., [126,195,196]) each criterion represents a single point of the probability distribution, so a better description of the distribution requires a higher number of criteria.

3.3 System for selecting the portfolio that maximizes confidence intervals

In order to evaluate the performance of our proposal to manage uncertainty, we first present the system in charge of exploiting the idea. Given that the Portfolio Optimization Problem is combinatorial, the system presented in the following subsections is based on Genetic Algorithms, which generally have good performance in these kinds of problems (see Subsection 2.4.2).

Algorithm

Considering several confidence intervals as evaluation criteria leads to a multicriteria optimization problem that can be addressed using a multiobjective Evolutionary Algorithm. We use MOEA/D in our system to find acceptable solutions to Problem (3.2.1). Nevertheless, even when MOEA/D is widely recognized as the most prominent MOEA based on aggregation of criteria, it has a poor diversity when dealing with instances having complicated PFs [175]. To overcome this shortcoming, we use some improvements introduced by Li and Zhang in Ref. [175]; namely, the setting of a maximal number of solutions replaced by each offspring solution, a selection of parents involving not necessarily only the neighborhood of the candidate solution, and a crossover that involves more than two parents. By using these new mechanisms, the exploration ability of the search can be improved. Moreover, an enhancement of the original algorithm is performed in order to deal with parameters defined as interval numbers. Algorithm 2 shows the procedure proposed to select portfolios characterized by confidence intervals. This algorithm is inspired on Refs. [175,295].

Since it is often computationally expensive to find the exact ideal point z^*, we use z, which is initialized in Step 3 and updated in Step 8 of the algorithm, as a substitute for z^* in (2.4.7). Furthermore, we use a nadir point, z^{nad}, to perform the normalization. Given that our implementation of MOEA/D has to deal with interval numbers instead of real numbers, we consider the lower and upper bound of the intervals to define z^{nad} and z, respectively. We update this reference through the lowest and highest possible value attainable by the confidence interval. Finally, the individuals belonging to the population at the last generation are considered as the result of one run. The individuals generated in 20 runs are introduced in a pool, from where the non α-dominated solutions are selected as the final approximation to the PF (cf. Section 2.3.4 to see the definition of α-dominance).

Chromosome representation

In this work, the chromosomes or individuals (alternatives of solution) in the population are represented by a string of n real numbers; that is, each gene in the chromosome is a real number. This is a common way of representing portfolios given its practicality in the representation of the resources assigned to stocks. The i-th gene in the chromosome specifies the proportion of resources assigned to stock i. Individuals are represented by a string composed of n positions as shown in Figure 3.1.

Figure 3.1: Individual encoding

x_1	x_2	$\cdots$	x_{n-1}	x_n

Algorithm 2 Algorithm proposed to address Problem (3.2.1)

Require: Problem (3.2.1), see Section 3.4 for an illustrative application; 100 generations as the stopping criterion; $n_r = 2$, the maximal number of solutions replaced by each offspring solution; $\delta = 0.9$, probability of selecting parents only from the neighborhood (instead of the whole population); $N = 100$, the number of sub-problems; $T = 20$, the number of weight vectors in the neighborhood of each weight vector.

Ensure: Approximation to the PS, $\{x^1, x^2, \cdots, x^N\}$; approximation to the PF, $\{\theta(x^1), \theta(x^2), \cdots, \theta(x^N)\}$.

1: Work out the T closest weight vectors to each weight vector. (Recall that a weight vector is a vector $\lambda^i = (\lambda_1^i, \cdots, \lambda_k^i)^\top$ that allows to weigh the k criteria in the i-th sub-problem and satisfies $\lambda_j^i \geq 0$ for all $j = 1, \cdots, k$ and $\sum_{j=1}^k \lambda_j^i = 1$.) For each $i = 1, \cdots, N$, set $B(i) = \{i_1, \cdots, i_T\}$ where $\lambda^{i_1}, \cdots, \lambda^{i_T}$ are the closest weight vectors to λ^i.

2: Generate an initial population $x^1, x^2, \cdots, x^N$ by uniformly randomly sampling from Ω. Set $FV^i = \theta(x^i)$ for $i = 1, \cdots, N$.

3: Initialize $z^{nad} = (z_1^{nad}, \cdots, z_k^{nad})^\top$ by setting $z_j^{nad} = \min_{1 \leq i \leq N} \alpha_j^i$, and $z = (z_1, \cdots, z_k)^\top$ by setting $z_j = \max_{1 \leq i \leq N} \beta_j^i$. Where α_j^i and β_j^i are the lowest and highest attainable return of solution i in the j-th criterion; that is, $\theta_j(x^i) = [\alpha_j^i, \beta_j^i]$. z^{nad} and z are used in the update step in order to normalize the fitness values of the criteria.

4: **for** $i = 1, \cdots, N$, **do**

5: **Selection of Mating/Update Range**: Define the population P, from where the offspring will be produced, as the neighborhood of λ^i (with a probability of δ) or as the whole population (with a probability of $1 - \delta$): uniformly randomly generate a number $rand$ from [0, 1], then set

$$P = \begin{cases} B(i) & \text{if } rand < \delta, \\ \{1, \cdots, N\} & \text{otherwise.} \end{cases}$$

6: **Reproduction**: Set $r_1 = i$ and randomly select two indexes r_2 and r_3 from P, then generate a solution $\hat{y}$ from x^{r_1}, x^{r_2} and x^{r_3} using genetic operators, finally perform a mutation operation on $\hat{y}$ with probability $p_m = 0.01$ to produce a new solution y.

7: **Repair**: If an element of y is out of the boundary of Ω, go to step 5.

8: **Update of z^{nad} and z**: For each $j = 1, \cdots, k$, if $z_j^{nad} > \alpha_j$, then set $z_j^{nad} = \alpha_j$; and if $z_j < \beta_j$, then set $z_j = \beta_j$, where $\theta_j(y) = [\alpha_j, \beta_j]$.

9: **Update of Solutions**: Set $c = 0$ and do the following:

1. If $c = n_r$ or P is empty, go to Step 10. Otherwise, randomly pick an index i from P.
2. Normalize $\theta_j(x^i) = [\alpha_j^i, \beta_j^i]$ for $j = 1, \cdots, k$, such that $\theta_j^{norm}(x^i) = [\alpha_j^{i,norm}, \beta_j^{i,norm}]$: Make $\alpha_j^{i,norm} = \frac{\alpha_j^i - z_j^{nad}}{z_j - z_j^{nad}}$ and $\beta_j^{i,norm} = \frac{\beta_j^i - z_j^{nad}}{z_j - z_j^{nad}}$.
3. Calculate $g(x^i|\lambda^i)^{norm} = \max_{1 \leq j \leq k}\{[1 - \beta_j^{i,norm}, 1 - \alpha_j^{i,norm}]\lambda^i\}$.
4. If $p(g(y|\lambda^i)^{norm} \geq g(x^i|\lambda^i)^{norm}) \geq 0.5$ then set $x^i = y$, $FV^i = \theta(y)$ and $c = c + 1$.
5. Remove i from P and go to 1.

10: **Stopping Criterion**: If the stopping criterion is satisfied, namely the number of iterations is 100, then stop and output $\{x^1, x^2, \cdots, x^N\}$ and $\{F(x^1), F(x^2), \cdots, F(x^N)\}$. Otherwise go to Step 4.

11: **end for**

Selection

Since $f(x|\lambda^j, z^*)$, as defined in Equation (2.4.7), is continuous on λ, the optimal solutions of the neighboring sub-problems should be close in the decision space. MOEA/D exploits the neighborhood relationship among the sub-problems for making its search effective and efficient [295]. Nevertheless, as stated above, MOEA/D shows poor diversity in its solutions when facing complicated PFs [175]. One reason of this problem is that the maximal number of solutions replaced by an offspring solution could be as large as T, the neighborhood size. This implies that a single solution may replace most of the current solutions of its neighboring sub-problems. As a result, diversity in the population could be significantly reduced. In this work we intend to overcome this limitation by, as was done in [175], letting the offspring solution replace no more than n_r solutions of the current population. Furthermore, the solutions replaced may not necessarily be in the neighborhood of the offspring solutions. But our proposal allows three different parent solutions to be randomly selected from the whole population with a probability of $1 - \delta$. The values T, n_r and δ are 20, 2 and 0.9, respectively, as in [175].

Crossover

We create one offspring solution from the information contained in the three parents selected. The crossover procedure works as follows. Let qG_1, qG_2, qG_3 be the quantity of genes satisfying $x_i > 0$ in parent 1, parent 2 and parent 3, respectively. The idea is that the parents provide similar proportions of gene material to the offspring. So, the number of genes satisfying $x_i > 0$ in the offspring solution is up to $qG_C = \frac{qG_1+qG_2+qG_3}{3}$ and each parent gives $\frac{qG_C}{3}$ randomly chosen genes to the offspring solution.

Mutation

The mutation operation simply consists in swapping two randomly chosen genes of the offspring solution. With the intention of a further improvement in the search exploration phase, the probability of mutation is $p_m = 0.01$.

Repairing process

In Section 2.4.2, we mentioned that the DM can consider several types of constraints during the optimization process, depending on her own preferences. The illustrative application shown in the next section considers only three of these constraints; namely, the budget constraint, the non-negativity constraint and the bounds on individual stocks constraint. We can ensure fulfillment of the last two constraints easily in the chromosomes construction (e.g., gene j is randomly selected in $[l_j, u_j]$ if $l_j \geq 0$). Nevertheless, the fulfillment of the budget constraint is not straightforward. The following techniques have been revised here with this goal (cf. [60,118]).

- Discard infeasible solutions. Given that our approach has only few constraints that are not often hit, the simplest approach is to "throw away" new infeasible solutions. That is, if a solution violates a constraint, we just select another one.
- Normalization. We can introduce mechanisms to correct solutions that violate constraints. For example, dividing every element in x by the sum of the elements of x ensures that all weights sum to unity.
- Ordering. Add the value of the ordered elements of x until the sum, σ, is greater or equal to one. Assign to the last element considered in the previous sum the value $\sigma - 1$. Finally, assign zero to the elements not considered in the sum.
- Penalization. Whenever a constraint is violated, we add a penalty term to the objective function and we consequently degrade the quality of the solution.

In preliminary experiments, we have found that simply discarding the infeasible solutions is the most suitable method in terms of time performance and quality of the solutions. Hence, this is the method we use to satisfy the constraints in the experiments shown below.

Fitness evaluation

As stated above, the Tchebycheff method is used to aggregate the criteria (see Subsection 2.4.2), its computation is given in Equation (2.4.7). In Step 9 of the algorithm we use this aggregation as the fitness of the solutions. In order to estimate the value of each criterion before the aggregation, a Montecarlo simulation is performed. This simulation allows us to find an approximation to the probability distribution of a given portfolio's return.

A simulation point consists in the random generation of the portfolio's return. This return is calculated as the weighted sum of the return of the stocks in the portfolio. Whereas the return of a stock is generated also in a random process where the historical returns of the stock are sampled. The "actual" return of the stock is randomly generated from a sample of the historical returns of the stock, where the probability of obtaining the actual return is given by the sample. The distribution of one thousand simulation points is assumed to be the real probability distribution of the return of the portfolio, and the confidence intervals are taken from this distribution according to the preferences expressed by the DM (Section 3.2).

A pseudo-random numbers generator known as Mersenne Twister [198] is used in the simulation. The algorithm of the generator has the following characteristics [199]:

- A period of $2^{19,936}$.
- An equidistribution property of 623 dimensions.
- Quick generation. (Although dependent of the system architecture, the authors report that MT is sometimes faster than the ANSI-C standard library.)

Final Selection

The number of solutions in the generated approximation to the PF may still be high enough to make the decision difficult. Thus, a final selection procedure needs to be performed.

Let portfolios x and y be two points in the PF of Problem (3.2.1). Furthermore, assume that[1]
$\theta_{70}(x) = [0.0171, 0.0479]$, $\theta_{99}(x) = [-0.1067, 0.0553]$, and
$\theta_{70}(y) = [-0.0004, 0.0220]$, $\theta_{99}(y) = [-0.0428, 0.0311]$.

Hence, $p(\theta_{70}(x) \geq \theta_{70}(y)) = 0.91$ and $p(\theta_{99}(x) \geq \theta_{99}(y)) = 0.42$. Although both x and y are non α-dominated (in the sense described in Section 2.3.4), portfolio x is arguably better than portfolio y. The final selection procedure followed here is based on the previous argument: let A be the set of solutions' performances in the approximation to the PF, similarly to the α-dominance concept described in Section 2.3.4, we say that x is non-dominated in A with degree β if and only if[2] $\min_{y \in A-\{x\}}\{\max_{1\leq j\leq k}\{p(\theta_{\gamma_j}(x) \geq \theta_{\gamma_j}(y))\}\} = \beta$. Our idea to exploit the risk management proposal is to select the portfolio that maximizes β.

3.4 Validating the uncertainty management proposal

Our proposal to manage uncertainty is applied here to the Portfolio Optimization Problem considering a risky objective, maximization of return. We argue that the uncertainty involved in this objective can be

[1]Portfolios x and y are two actual portfolios obtained in the experiments below.

[2]Here, we consider the *minimum* to represent conjunction, and the *maximum* to represent disjunction. In this sense, $\max_{1\leq j\leq k}\{p(\theta_{\gamma_j}(x) \geq \theta_{\gamma_j}(y))\}$ is interpreted as the credibility of x being non-dominated by y. Then, $\min_{y\in A-\{x\}}\{\max_{1\leq j\leq k}\{p(\theta_{\gamma_j}(x) \geq \theta_{\gamma_j}(y))\}\}$ is interpreted as the credibility of x being non-dominated by any solution in the PF.

encompassed by probabilistic confidence intervals, and use the system described above to exploit such idea.

The selection of financial portfolios refers to the analysis of financial objects (e.g., stocks, funds, bonds) to allocate resources that maximize the impact on the objectives of the decision maker. We show now an application on the allocation of resources among a set of stocks with the risky objective of maximizing the return.

3.4.1 Selecting confidence intervals

Stock portfolio selection consists of two stages (see [159,287,312]): stock valuation and portfolio optimization. The first stage chooses "the best" subset of stocks, while the second stage assigns a proportion of money to each of the chosen stocks. Here, we focus on the second stage.

Following Section 3.2, once the approximation to the return's probability distribution has been constructed, we can obtain as many confidence intervals as needed. For this illustrative example, we simulate a highly risk-averse DM that requests information on two intervals, one containing the portfolio return with 70% of probability and the other containing it with the 99% of probability. Thus, the system presented in Subsection 3.3 must address the following problem:

$$\underset{x\in\Omega}{\text{maximize}}(\{\theta_{70}(x), \theta_{99}(x)\}), \tag{3.4.2}$$

subject to

$\sum x_j = 1 \longrightarrow$ Budget constraint;

$x_j \geq 0 \longrightarrow$ Non-negativity conditions;

$x_j \leq 0.4 \longrightarrow$ Bounds on individual stocks;

$(j = 1, \cdots, n)$.

Where

$\theta_{70}(x) = \{[\alpha_{70}, \beta_{70}] : \mathbb{P}(\alpha_{70} \leq R(x) \leq \beta_{70}) = 0.70\}$, and

$\theta_{99}(x) = \{[\alpha_{99}, \beta_{99}] : \mathbb{P}(\alpha_{99} \leq R(x) \leq \beta_{99}) = 0.99\}$.

Later, we compare the solutions to Problem (3.4.2) with the solutions obtained by a less risk-averse DM. Thus, we now simulate a lowly risk-averse DM that requests information on intervals with the 30% and 50% of probability to contain the actual portfolio return. For this case, the proposed approach must solve the problem given by

$$\underset{x\in\Omega}{\text{maximize}}(\{\theta_{30}(x), \theta_{50}(x)\}), \tag{3.4.3}$$

subject to

$\sum x_j = 1 \longrightarrow$ Budget constraint;

$x_j \geq 0 \longrightarrow$ Non-negativity conditions;

$x_j \leq 0.4 \longrightarrow$ Bounds on individual stocks;

$(j = 1, \cdots, n)$.

Where

$\theta_{30}(x) = \{[\alpha_{30}, \beta_{30}] : \mathbb{P}(\alpha_{30} \leq R(x) \leq \beta_{30}) = 0.30\}$, and

$\theta_{50}(x) = \{[\alpha_{50}, \beta_{50}] : \mathbb{P}(\alpha_{50} \leq R(x) \leq \beta_{50}) = 0.50\}$.

3.4.2 Experimental design

Dataset

We describe in this section the dataset used in our experiments. In these experiments, the performance of our proposal is compared with that of a highly important market index, namely, the Dow Jones Industrial Average, DJIA. The DJIA index contains the stocks of 30 of the largest companies in the United States of America.

Following [267], the main contraindication of using market indexes as benchmarks is that the profitability of portfolios is often compared to popular indexes such as DJIA, regardless of portfolio size or classification of its stocks. Most investors expect to reach or exceed the yields of these indexes over time. The problem with this expectation is that they are at a disadvantage because they are not "comparing apples to apples". That is, there is no guarantee that the characteristics of the stocks in the portfolio coincide with the characteristics of the stocks contained in the index. We avoid this trap by incorporating into the portfolio only the stocks of the index being considered as benchmark.

We use the historical monthly returns of the stocks in the DJIA index for the period April 1998-March 2016 (see e.g., [287]) to perform a back-testing strategy (cf. [220]). Each investment horizon goes from April of the current year to March of the following year because the yearly financial information is publicly available for the stock market in March [181]. The reason for using this particular period is because the index shows upward and downward trends, so there are multiple scenarios to validate our proposal. The duration of the period is an approximate average of the horizons used in several articles of the literature revised. Finally, similarly to Refs. [124,181], we use a sliding time window of 60 months/1 month. That is, we use five years for model training (e.g., we obtain metrics of the data set from April 1998 to March 2003) and one month for validation (e.g., we use the metrics obtained to create a portfolio and estimate its monthly performance in April 2003). The process is then repeated for each period of one month (in a sliding window manner) until the end of the evaluation period (see e.g., [124]). In other words, we consider a buy and hold strategy (B&H), where we select the best stock portfolio of the current month by solving Problem (3.4.2) or Problem (3.4.3) and using the historical metrics of the previous five years. This portfolio is maintained over a one-month investment horizon. Each time we start a new investment horizon, we review the stock portfolio (i.e., select a new distribution of resources among the stocks) according to the corresponding horizon's valuation.

As done in other works (see e.g., [10,56,124,187,302]), the historical monthly prices of the stocks and index were downloaded from the Yahoo! Finance database [288]. DJIA index updated its listed stocks several times during the period considered. Thus, the data retrieving process starts by finding out the corresponding stocks to a specific year. The configuration of the historical data downloaded from the database is *Date*, *Open*, *High*, *Low*, *Close*, *Volume*, and *Adj. Close*. We use the *Close* parameter to calculate the returns. All data used in this work is available for consultation upon request.

3.4.3 Results

We show in this section the evaluation results when validating the capacity of our proposal to manage uncertainty. First, we provide the results obtained when solving Problem (3.4.2); that is, when the DM is highly risk averse. Later, we show the results obtained when solving Problem (3.4.3); that is, when the DM is lowly risk averse. For both situations, we compare the solutions of the proposed approach using three benchmarks: the DJIA index (Subsection 2.1.4.3), the mean-variance model (Subsection 2.1.1), and the results provided by Gorgulho et al., in Ref. [124]. The latter comparison is valid given that the dataset used in that work is a subset of the one used in this paper.

3.4.3.1 Selecting portfolios with high risk aversion

Results discussed in this subsection correspond to the assessment of solutions generated by our proposal when addressing Problem (3.4.2).

Comparing with Dow Jones Industrial Average index Tables 3.2 and 3.3 show the portfolios, obtained by our proposal, that produce the most extreme returns when solving Problem (3.4.2); the worst return, obtained for February 2009, and the best return, obtained for July 2009. Both tables show the stocks in the portfolio, the actual return of these stocks and the proportion of resources assigned by the proposed approach to each stock. Finally, both tables show also the return of the portfolio and the corresponding confidence intervals obtained in the simulation. The portfolio shown in Table 3.2 produces an actual return of $R(x) = -0.1243$, while its 70% confidence interval is $[0.0004, 0.0141]$ and its 99% confidence interval is [-0.0389, 0.0188]. Note how the actual return of the portfolio is far from being within the confidence intervals. This is due to the high volatility produced by the crisis. Interestingly, the worst return obtained by our approach was not during the market crisis of October 2008. The return of DJIA index in this month was -0.1406 (the lowest in the whole period considered) while the return obtained by the proposed approach was -0.0797. We believe this situation is due to a consistency in the losses of the stocks before October 2008. That is, the stocks of the DJIA index with the greatest losses (AA, AIG, CAT, ⋯) had presented highly negative returns before October 2008, making the confidence intervals of the portfolios containing those stocks to have low values and the approach to neglect most of them. It was not until February 2009 that the volatility and the lack of consistency in the historical returns of the stocks had repercussions on the performance of our proposal. But even when this was the worst performance of our proposal in the whole period, it was actually not too far from the return of the DJIA index in the corresponding month, -0.1172. The aggressive recovery of the stocks in the following months allowed the proposed approach to find its best performance of the entire period, 0.0986, in April 2009. This return was greater than each return produced by the index in the thirteen years.

Table 3.2: Portfolio created by the proposed approach for February 2009 when solving Problem (3.4.2). Its return corresponds to the lowest return obtained in the whole period 2003-2016.

$R(x) = -0.1243$ $x_{70} = [0.0004, 0.0141]$ $x_{99} = [-0.0389, 0.0188]$		
Stock	**Return**	x_i
Alcoa Corp (AA)	-0.2003	0
American International Group Inc (AIG)	-0.6719	0
American Express Company (AXP)	-0.2791	0
Boeing Co. (BA)	-0.2569	0
Bank of America Corporation (BAC)	-0.3997	0
Citigroup, Inc (C)	-0.5775	0
Caterpillar Inc. (CAT)	-0.2023	0
Chevron Corporation (CVX)	-0.1391	0.13
EI du Pont de Nemours & Co (DD)	-0.1829	0
Walt Disney Company (DIS)	-0.1891	0

Continued on next page

Table 3.2 – *Continued from previous page*

Stock	Return	
General Electric Company (GE)	-0.2984	0
Home Depot, Inc. (HD)	-0.0297	0
HP Inc. (HPQ)	-0.1646	0.198
International Business Machines Corporation (IBM)	0.0041	0
Intel Corporation (INTC)	-0.0124	0
Johnson & Johnson (JNJ)	-0.1333	0
JPMorgan Chase & Co. (JPM)	-0.1043	0
Coca-Cola Company (KO)	-0.0438	0
McDonald's Corporation (MCD)	-0.0994	0.316
3M Co. (MMM)	-0.1549	0
Merck & Co., Inc. (MRK)	-0.1524	0.056
Microsoft Corporation (MSFT)	-0.0556	0
Pfizer Inc. (PFE)	-0.1557	0
Procter & Gamble Co. (PG)	-0.1161	0
AT&T Inc. (T)	-0.0345	0
United Technologies Corporation (UTX)	-0.1492	0
Verizon Communications Inc. (VZ)	-0.0449	0
Wal-Mart Stores Inc. (WMT)	0.0450	0
Exxon Mobil Corporation (XOM)	-0.1122	0.3

Table 3.3: Portfolio created by the proposed approach for July 2009 when solving Problem (3.4.3). Its return corresponds to the highest return obtained in the whole period 2003-2016.

$R(x) = 0.0986$

$x_{70} = [-0.0214, 0.0488]$ $x_{99} = [-0.1024, 0.1522]$

Stock	Return	x_i
Apple Inc. (AAPL)	0.1384	0
American Express Company (AXP)	0.219	0
Boeing Co. (BA)	0.0096	0
Caterpillar Inc. (CAT)	0.1205	0
Cisco Systems, Inc. (CSCO)	0.3335	0.192
Chevron Corporation (CVX)	0.1802	0
EI du Pont de Nemours & Co (DD)	0.0486	0
Walt Disney Company (DIS)	0.2073	0
General Electric Company (GE)	0.0767	0

Continued on next page

Table 3.3 – *Continued from previous page*

Stock	Return	
Goldman Sachs Group Inc. (GS)	0.1433	0
Home Depot, Inc. (HD)	0.0978	0
International Business Machines Corporation (IBM)	0.1203	0.395
Intel Corporation (INTC)	0.1294	0
Johnson & Johnson (JNJ)	0.1631	0
JPMorgan Chase & Co. (JPM)	0.072	0.039
Coca-Cola Company (KO)	0.1331	0
McDonald's Corporation (MCD)	0.0385	0
3M Co. (MMM)	-0.0423	0.374
Merck & Co., Inc. (MRK)	0.1734	0
Microsoft Corporation (MSFT)	0.0733	0
Nike Inc. (NKE)	-0.0105	0
Pfizer Inc. (PFE)	0.062	0
Procter & Gamble Co. (PG)	0.0863	0
Travelers Companies Inc. (TRV)	0.056	0
UnitedHealth Group Inc. (UNH)	0.0495	0
United Technologies Corporation (UTX)	0.0483	0
Visa Inc. (V)	0.0436	0
Verizon Communications Inc. (VZ)	0.0297	0
Wal-Mart Stores Inc. (WMT)	0.0069	0
Exxon Mobil Corporation (XOM)	0.1384	0

Figure 3.2 shows the returns obtained by our proposal and by the Dow Jones Industrial Average index in the period 2003-2016. The difference between these results is shown in Figure 3.3. Although this figure shows that there are several occasions where the difference is against the proposal (bars below zero), the number of times and magnitude of difference when the proposed approach outperforms the index is greater. Figure 3.4 confirms this through the accumulative return. Recall that the allocation of resources is performed on a monthly basis (the returns are obtained at the end of the month and added to the cumulative sum; later, the portfolio is reconfigured and a new allocation is performed), which implies that each of these figures actually provides comparisons between the proposal and the reference index on the basis of 156 scenarios.

It is interesting to highlight that the decrease of the proposed approach's accumulative return in the period Mar/2008-Feb/2009 is 0.27, while the decrease of the DJIA index in the same period is 0.52. This implies that our proposal was more robust than the DJIA index in the worst crisis of the last years.

Figure 3.2: Returns produced in the period 2003-2016 by the DJIA index and the proposed approach when solving Problem (3.4.2)

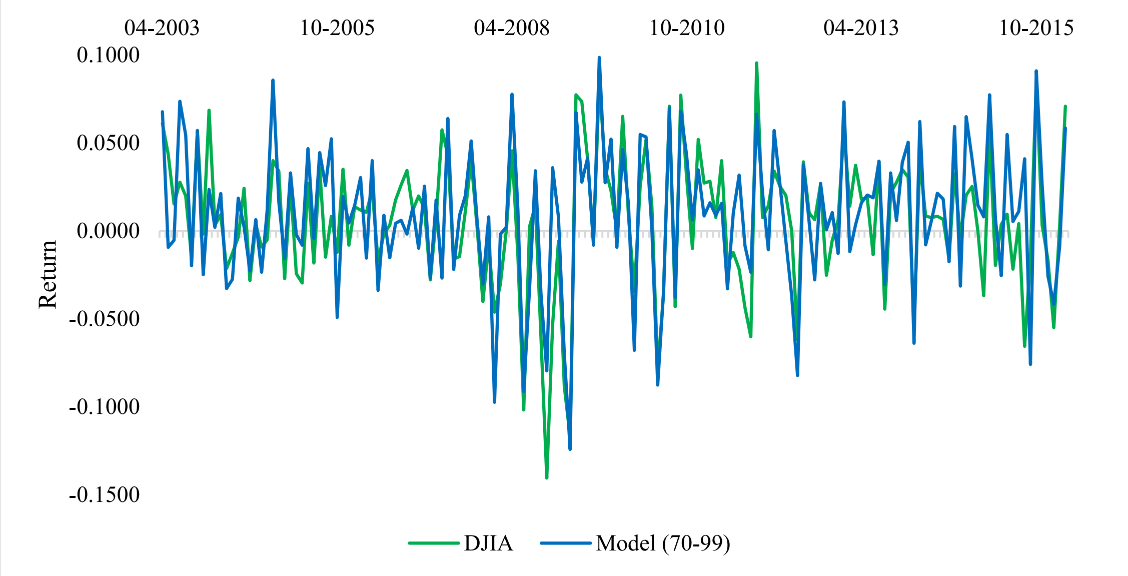
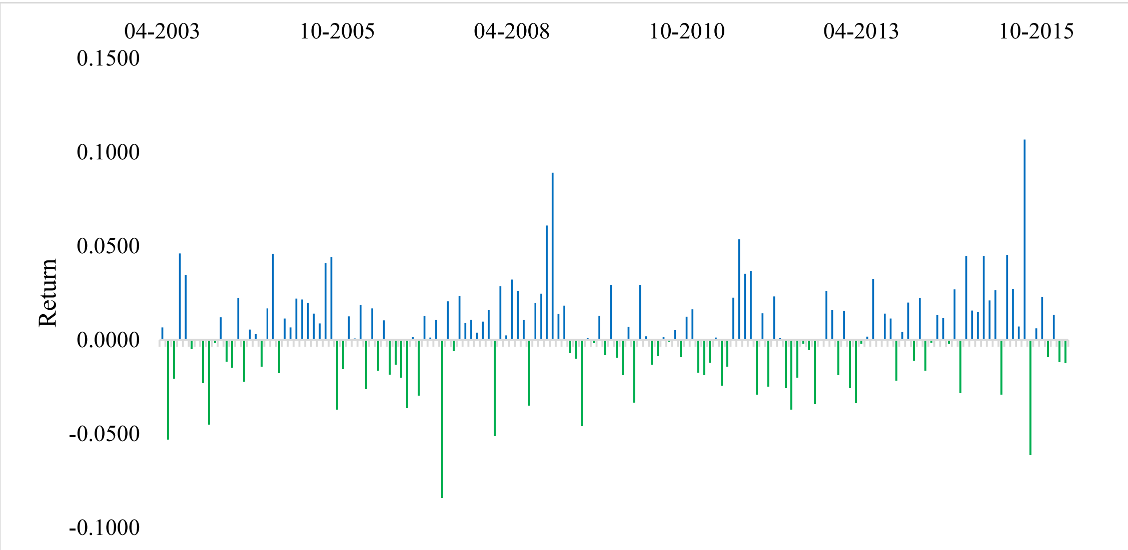


Figure 3.3: Difference of the returns obtained by the proposed approach when solving Problem (3.4.2) and the DJIA index in the period 2003-2016



Aiming to analyze the quality of our approximation to the PF of Problem (3.4.2), we show the performance of its extremes (i.e., the portfolio that maximizes criterion θ_{99} and the portfolio that maximizes criterion θ_{70}) and the average of its solutions' performances. Figure 3.5 shows i) the accumulative return of the DJIA index (identified as DJIA); ii) the accumulative return of the portfolio maximizing criterion θ_{70} (*Model (70)*); iii) the accumulative return of the portfolio maximizing criterion θ_{99} (*Model (99)*); and iv) the average accumulative return of the portfolios in the PF (*Model (Average)*).

Figure 3.4: Accumulative return produced in the period 2003-2016 by the DJIA index and the proposed approach when solving Problem (3.4.2)

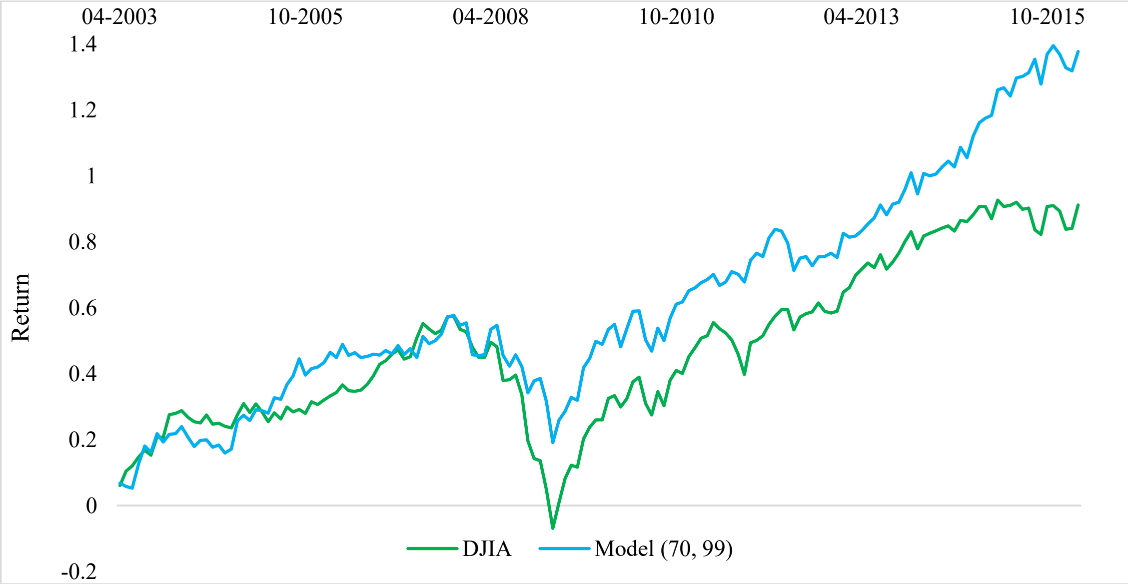
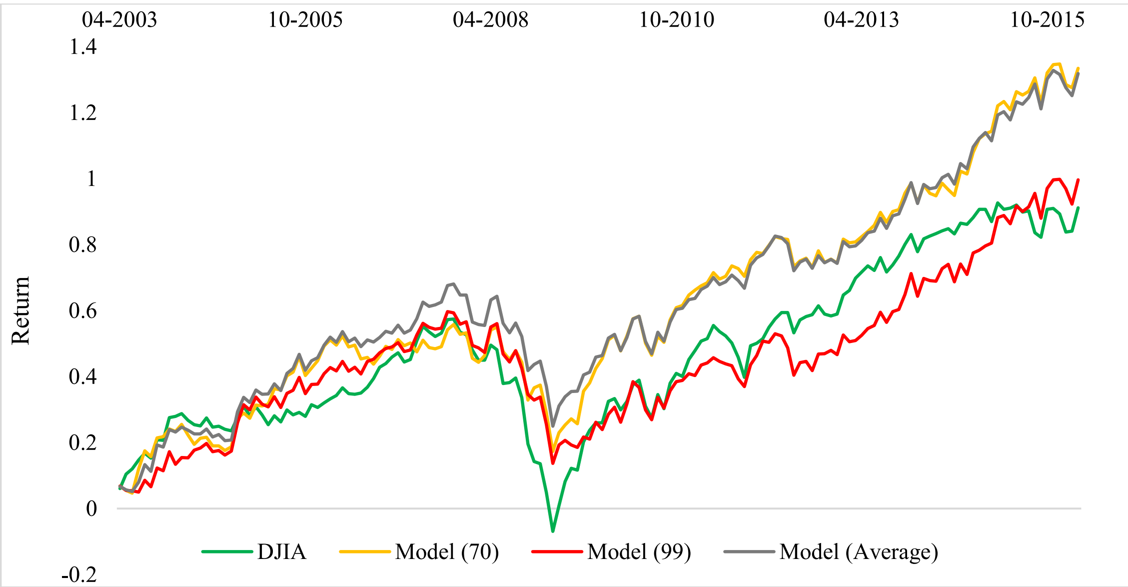


Figure 3.5: Description of the PF obtained by the proposed approach when solving Problem (3.4.2)



Comparing with the mean-variance model Now, we compare the results of the proposal with those of the mean-variance (MV) model [195]. Figure 3.6 shows the comparison. In this Figure, the approximation to the PF made by the MV *classical formulation* in a given month is described using the average of all the returns within the whole approximation for that month (*MV (average)*). We also used the *risk aversion formulation* and, similarly to Ref. [69], defined the high-risk aversion parameter as $\gamma = 4$. (See Subsection 2.1.1 for the definition of both formulations.)

Figure 3.6: Accumulative return produced in the period 2003-2016 by the Mean-Variance model and the proposed approach when solving Problem (3.4.2)

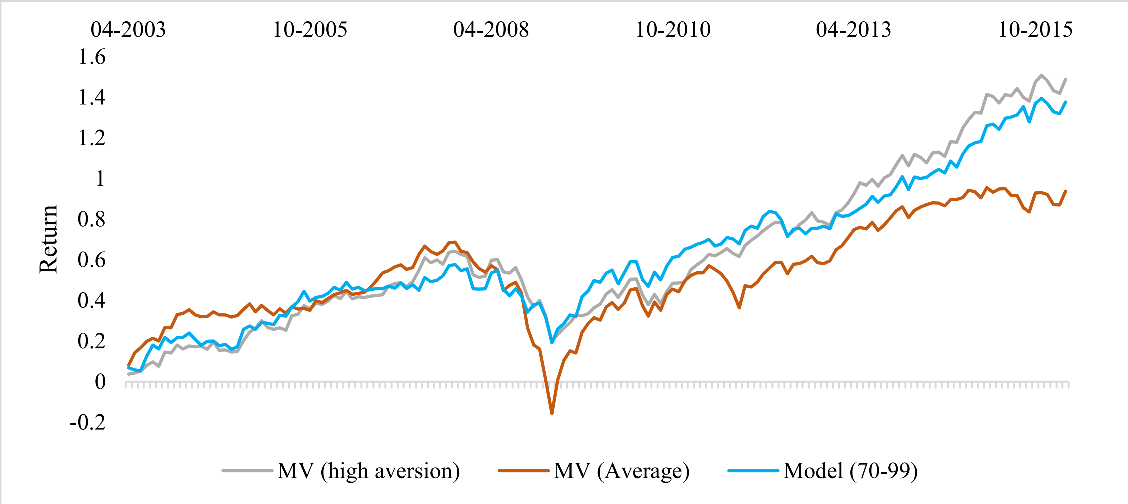


Figure 3.7: Accumulative return shown in the period 2003-2009 by Ref. [124] and the proposed approach when solving Problem (3.4.2)

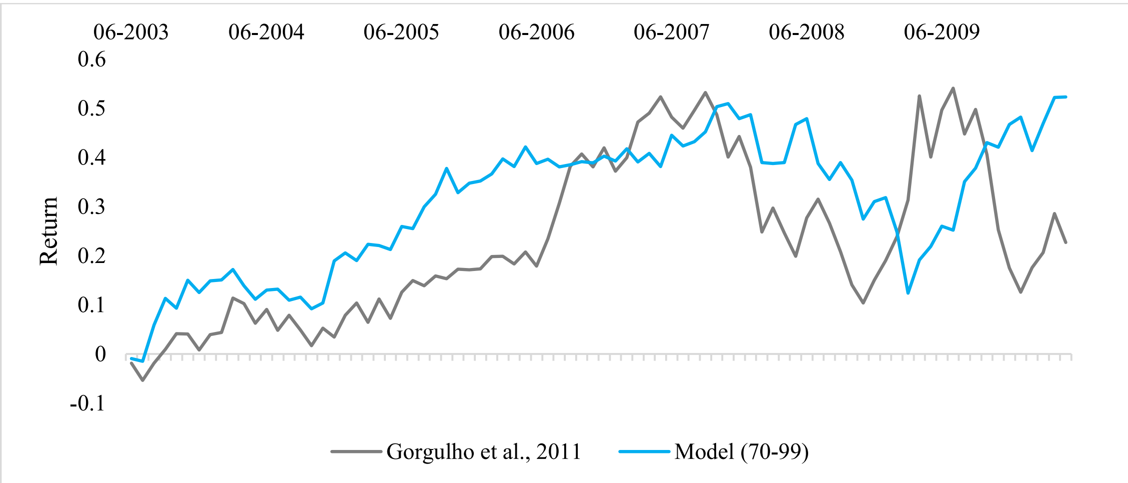


Figure 3.6 shows superiority of the proposed approach over the mean-variance when using the classical formulation. Nevertheless, it ends up being outperformed by the mean-variance when using the risk aversion formulation. An interesting result shown in this figure is the fall suffered by the mean-variance model during the 2008 crisis. In this period, its fall is appreciably steeper than that of our proposal. This, together with the also steeper rise of the mean-variance model, might indicate lack of representativeness of the DM's risk behavior.

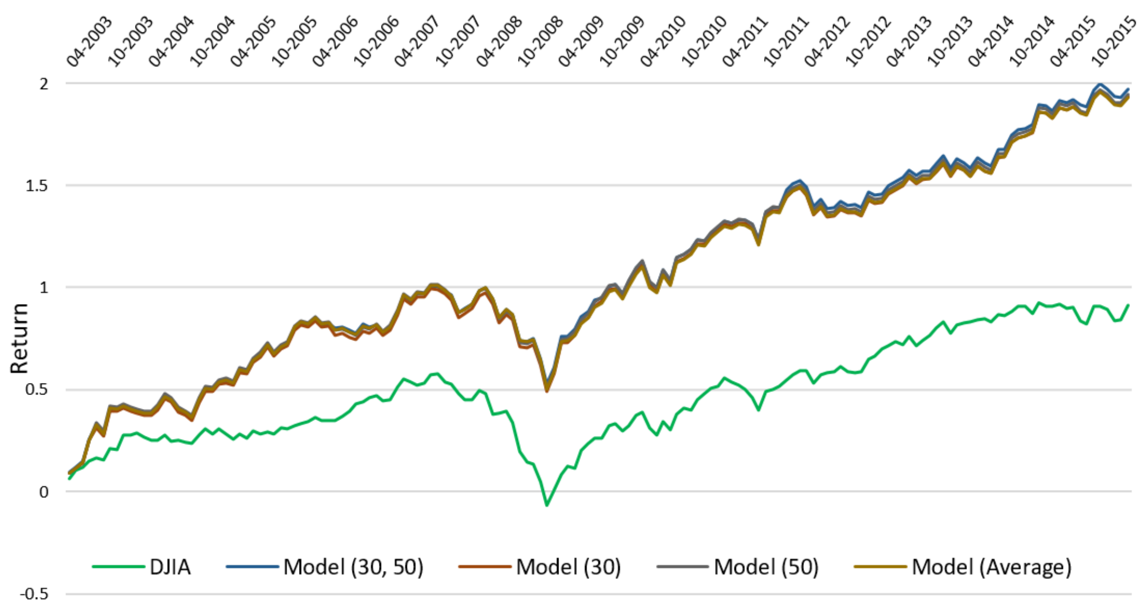
Comparing with a recent benchmark Finally, we compare the performance of our proposal with that of a recently published work [124] whose dataset is a subset of the one used here. Particularly, they used the returns of stocks within the DJIA index in the period 2003-2009. Hence, the comparison is in that specific period. The comparison of the results is shown in Figure 3.7.

3.4.3.2 Selecting portfolios with low risk aversion

Results discussed in this subsection correspond to the assessment of solutions generated by our proposal when addressing Problem (3.4.3).

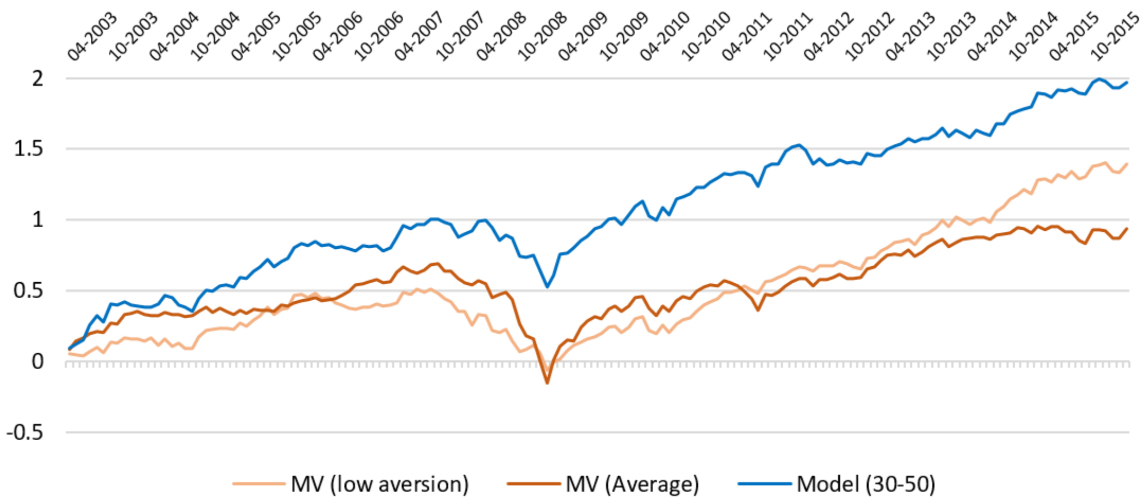
Comparing with Dow Jones Industrial Average index Figure 3.8 shows i) the accumulative return of the DJIA index (identified as DJIA); ii) the accumulative return of the portfolio with the highest non-dominance degree from the PF, (*Model (30, 50)*); iii) the accumulative return of the portfolio maximizing criterion θ_{30} (*Model (30)*); iv) the accumulative return of the portfolio maximizing criterion θ_{50}, (*Model (50)*); and v) the average accumulative return of the portfolios in the PF, (*Model (Average)*).

Figure 3.8: Performance of some solutions in the PF obtained by the proposed approach when solving Problem (3.4.3)



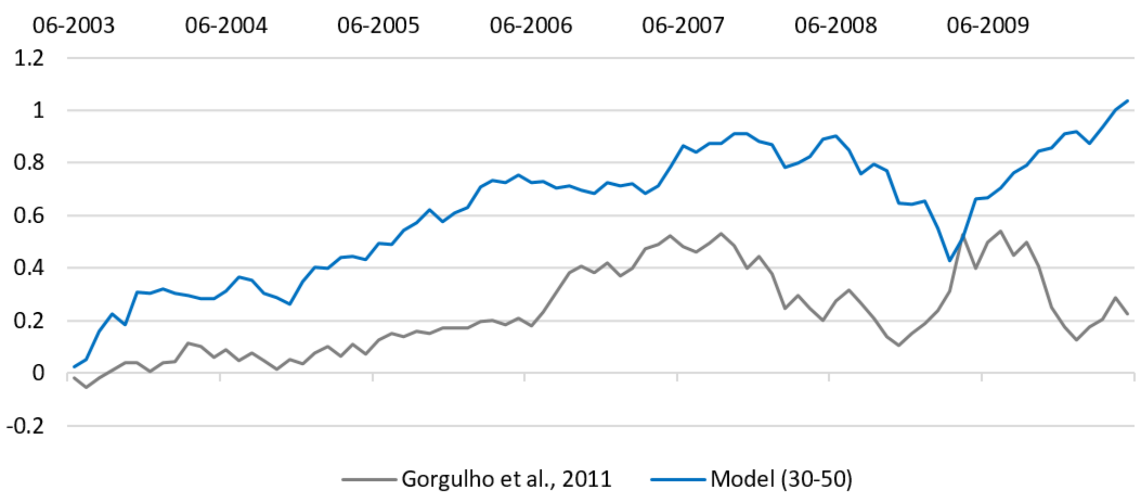
Comparing with mean-variance model Figure 3.9 presents a comparison between the performance of the solutions found by the proposal when solving Problem (3.4.3) and the mean-variance model in its classic and risk aversion formulations. Here, similarly to Ref. [69], we defined the low risk aversion as $\gamma = 3$.

Figure 3.9: Accumulative return produced in the period 2003-2016 by the Mean-Variance model and the proposed approach when solving Problem (3.4.3)



Comparing with a recent benchmark Figure 3.10 shows the comparison between the solutions of the proposal and the results provided by Ref. [124].

Figure 3.10: Accumulative return shown in the period 2003-2009 by Ref. [124] and the proposed approach when solving Problem (3.4.3)



3.4.4 Discussion

It is evidently that the performance of the approach when solving Problem (3.4.3) is better than the performance of the approach when solving Problem (3.4.2). This situation is due to the general uptrend of the market and the conservativism of the approach when solving Problem (3.4.2) that prevents it from taking advantage of the trend.

In order to support this claim, we develop an analysis of the last year of the crisis (Mar/2008-Feb/2009) and the subsequent year to the crisis (Feb/2009-Jan/2010) that allows us to see the effects of risk aversion embodied by both problems. We selected these years because they present the steepest fall and rise of the whole period.

From March 2008 to February 2009,

- the accumulative return of the portfolio from the PF that maximizes the 99% confidence interval fell from 0.47 to 0.14 (a difference of 0.33);
- the accumulative return of the portfolio from the PF that maximizes the 70% confidence interval fell from 0.46 to 0.17 (a difference of 0.29);
- the accumulative return of the portfolio from the PF that maximizes the 50% confidence interval fell from 0.91 to 0.51 (a difference of 0.40);
- the accumulative return of the portfolio from the PF that maximizes the 30% confidence interval fell from 0.90 to 0.49 (a difference of 0.41).

From February 2009 to January 2010,

- the accumulative return of the portfolio from the PF that maximizes the 99% confidence interval raised from 0.14 to 0.38 (a difference of 0.24);
- the accumulative return of the portfolio from the PF that maximizes the 70% confidence interval raised from 0.17 to 0.58 (a difference of 0.41);
- the accumulative return of the portfolio from the PF that maximizes the 50% confidence interval raised from 0.51 to 1.10 (a difference of 0.59);
- the accumulative return of the portfolio from the PF that maximizes the 30% confidence interval raised from 0.49 to 1.08 (a difference of 0.59).

(As a reference, the DJIA index fell 0.52 and raised 0.37.)

Hence, in these periods there was, although not clearly, a tendency to decrease losses in the downtrend as the probability of the intervals increases, and to increase profits in the uptrend as the probability of the intervals decreases. This indicates a correct modeling of the DM's attitude when facing risk.

We also see that, in general, the solutions with the best performance are those with the highest non-dominance degree. Finally, we can see that the performance of the portfolios generated by the approach, and particularly those generated by solving Problem (3.4.3), clearly outperforms not just the Dow Jones Industrial Average index but also the performance of some portfolios built by other researchers in the literature [124,136,155,195].

Chapter 4

An elicitation method of the decision maker's system of preferences

The results of the portfolio optimization must be in agreement with the decision maker's preferences. However, it is usually very difficult to obtain the values of the parameters in models representing the decision maker's system of preferences. This difficulty is a source of imprecision, uncertainty, ill-determination and arbitrariness. We describe and evaluate in this chapter our proposal to indirectly elicit such parameter values when the preference model in consideration is the interval-based outranking approach (Subsection 2.3.4).

The proposal is extensively assessed in its ability to reproduce the DM's preferences in two contexts. First, we evaluate the proposal's effectiveness to define the same binary preference relations between portfolios as the ones stated by the DM. Later, the proposal's effectiveness to produce the same assignment of portfolios to categories as the one made by the DM is evaluated. Results show a high effectiveness of the proposal in the two contexts both in-sample and out-of-sample, and with many criteria.

This chapter starts by providing an introduction to the indirect elicitation procedures and, particularly, the so-called Preference Disaggregation Analysis. Section 4.2 formalizes our proposal to elicit the parameter values of the interval-based outranking method; such parameter values represent an approximation to the decision maker's system of preferences. In order to evaluate the proposal, a genetic-algorithm-based system is presented in Section 4.3. Finally, an extensive evaluation is performed and described in Section 4.4.

4.1 Introduction

In most cases, the Portfolio Optimization Problem can be addressed by a multicriteria decision aiding (MCDA) method (cf. Section 1.1). MCDA methods are mostly related to an aggregation of conflicting criteria to reach a final decision or recommendation. As a consequence of its conflictive nature and unlike the decision making under a single criterion, decision making under multiple criteria prevents the existence of an ideal solution. Hence, the decision maker's (DM) particular system of preferences (decision policy) becomes the primary tool for choosing the most preferred solution from a mathematically equivalent set of solutions. Here, we assume that the DM has already made (or agrees with) a set of (reference) decisions; thus, his/her preferences have been implicitly aggregated and are present in such decisions. Our aim in this chapter is therefore to provide and evaluate a procedure that allows us to elicit the DM's decision policy through a disaggregation of the preferences in these decisions.

Portfolio Optimization Problem involves a finite set of portfolios (decision objects, potential actions, alternatives of solution); in this work, we consider the case where the DM needs to evaluate these portfolios under the selection and the ordinal classification problems (P_α and P_β problems [242], cf. Subsection 2.3.2). Multicriteria decision aiding provides a wide range of appropriate methodologies for such situations. However, the aid provided by MCDA is not effective unless it represents the decision policy of the DM with an acceptable accuracy and in a congruent manner with respect to the specific characteristics of the problem. This can be done through an interaction between the DM and a Decision Analyst (DA). However, there are situations where an interaction with the DM does not solve the problem, first because of the complexity of the problem and second because the DM may not be accessible (e.g., because he/she does not even exist). In those cases, an indirect method has to be used. When using an interactive communication session approach between the decision analyst and the decision maker, the DA obtains specific information about the DM's preferences (e.g., weights, thresholds, etc.). Nevertheless, it is possible that the number of parameters required by the decision aiding model is overwhelming and the DM cannot or will not commit to provide the information. In any case, the task is not easy, since the DM usually has difficulties to explicitly specify numerical parameters and the time and cognitive effort required to do so may be inhibitory. The methods of preference disaggregation [144] are useful in this context. Preference disaggregation methods analyze decisions made by the DM in order to identify the aggregation model that underlies the outcome of the known decision. The Preference Disaggregation Analysis (PDA) paradigm infers the decision-making parameters of the DM from holistic decisions provided by her/him and uses regression-like methods to produce a decision model as consistent as possible with the set of reference decisions.

The Preference Disaggregation Analysis paradigm is of growing interest because it implies a lower cognitive effort from the DM. Nevertheless, the concept of "true" value of a decision model parameter is ill-defined due to several reasons: i) the DM's decision policy does not match exactly with the model's assumptions and mathematical structure; b) the DM's decision policy is poorly-defined (e.g. a heterogeneous group); c) the DM is a mythical or inaccessible person, and d) poor information on criterion performances. Thus, there is always imprecision, uncertainty, ill-definition and/or arbitrariness (imperfect knowledge, according to Roy et al., [246]) to be handled by the PDA when eliciting the values of the parameters. It would be more convenient if, instead of punctual values, the indirect elicitation offered the flexibility to consider the parameters as ranges of numbers, where the imperfect knowledge is contained within the interval.

Many works in the literature have applied PDA to get a decision model consistent with the decision-maker's holistic decisions. To the best of our knowledge, however, all these studies have assumed the parameters of the underlying aggregation model as punctual values, even when there have been proposals (e.g., [97,178,179]) where the models' parameters are in fact being considered as ranges of values. The main contribution of the proposal described in this chapter is therefore to obtain decision model parameters consistent with a reference set when each parameter is represented as a range of numbers; particularly, we consider the case where the decision model adopted is the interval-based outranking approach (Subsection 2.3.4). Let us now present this proposal.

4.2 Formalization of the proposal

The imperfect knowledge that characterizes the decision maker's (DM) implicit model of preferences [246] rises the idea that vague or ill-determined information should be considered during the modeling of the DM's preferences [97]. However, it is often difficult for the DM to express specific values for the parameters of models representing her/his own preferences [216], even when these parameters are defined as ranges of numbers as described in Subsection 2.3.4.

In this chapter we present and validate a novel way to indirectly obtain the DM's implicit system of preferences through an interval-based Preference Disaggregation Analysis. The main characteristic of the proposal relies in allowing the DM's decision policy to contain imperfect knowledge. Such imperfect knowledge is modeled here in form of interval numbers. The inference rules used by our proposal are based on plausible assumptions that can be easily interpreted and specifically suited by the DM/analyst couple. Furthermore, the construction of the reference set is intended to avoid an arduous work of the DM by letting her/him simply assign some portfolios to the categories (also known in the related literature as classes), or just agree with such assignments. The DM's decision policy is reflected by such reference set, and it is from this set where the proposal draws an approximation to the DM's implicit system of preferences.

Let us now introduce/recall some assumptions that are basic for presenting our proposal:

1. There is a finite set $\mathscr{A}$ of portfolios described by a coherent family of criteria $\mathscr{F} = \{g_1(\cdot), \cdots, g_k(\cdot)\}$ (in the sense of [46]), where $g_j(x) = [g_j^-(x), g_j^+(x)]$ is the interval number that represents the performance evaluation of portfolio $x \in \mathscr{A}$ in attribute g_j; without loss of generality, it is assumed that all criteria are to be maximized.

2. The DM makes decisions on the basis of the *interval-based outranking decision model* (Subsection 2.3.4) described by the set of parameters $\mathscr{P} = \{w_1, \cdots, w_k, v_1, \cdots, v_k, \lambda, \beta_0\}$.

3. There is a finite set, $\mathbf{C} = \{C_1, \cdots, C_r\}$, of ordered categories, such that if $i > j$, then the DM prefers the elements in C_i over the elements in C_j.

4. For each pair $(x, y) \in \mathscr{A} \times \mathscr{A}$, the following preference relations are defined based on the likelihood index associated with "x is at least as good as y" defined in Subsection 2.3.4 and calculated on the basis of $\mathscr{P}$:

   - **Strict preference**: $xP_{\mathscr{P}}y \iff \beta(x, y) \geq \beta_0 \wedge \beta(y, x) < 0.5$,
   - **Weak preference**: $xQ_{\mathscr{P}}y \iff \beta(x, y) \geq \beta_0 \wedge 0.5 \leq \beta(y, x) < \beta_0$,
   - **K preference**: $xK_{\mathscr{P}}y \iff \beta(y, x) < 0.5 < \beta(x, y) < \beta_0$,
   - **Indifference**: $xI_{\mathscr{P}}y \iff \beta(x, y) \geq \beta_0 \wedge \beta(y, x) \geq \beta_0$,

   where $\wedge$ is the conjunction operator and $\beta(x, y)$ is the likelihood index of the assertion "x is at least as good as y", xSy.

5. The DM assigns each $x \in \mathscr{A}$ to one category $C_j \in \mathbf{C}$ on the basis of $\mathscr{P}$. (Alternatively, the DM accepts its assignation to the category.) We denote such assignment as $C_{\mathscr{P}}(x) = j$. For all $(x, y) \in \mathscr{A} \in \mathscr{A}$ the assignment should be consistent with the following conditions:

$$C_{\mathscr{P}}(x) - C_{\mathscr{P}}(y) \geq 2 \Longrightarrow xP_{\mathscr{P}}y, \tag{4.2.1}$$

$$C_{\mathscr{P}}(x) - C_{\mathscr{P}}(y) = 1 \Longrightarrow xP_{\mathscr{P}}y \vee xQ_{\mathscr{P}}y \vee xK_{\mathscr{P}}y, \tag{4.2.2}$$

$$C_{\mathscr{P}}(x) - C_{\mathscr{P}}(y) = 0 \Longrightarrow xI_{\mathscr{P}}y \vee xQ_{\mathscr{P}}y \vee yQ_{\mathscr{P}}x \vee xK_{\mathscr{P}}y \vee yK_{\mathscr{P}}x, \tag{4.2.3}$$

$$xP_{\mathscr{P}}y \Longrightarrow C_{\mathscr{P}}(x) > C_{\mathscr{P}}(y) \tag{4.2.4}$$

$$xQ_{\mathscr{P}}y \vee xK_{\mathscr{P}}y \Longrightarrow C_{\mathscr{P}}(x) \geq C_{\mathscr{P}}(y) \tag{4.2.5}$$

$$xI_{\mathscr{P}}y \Longrightarrow C_{\mathscr{P}}(x) = C_{\mathscr{P}}(y) \tag{4.2.6}$$

   where $\vee$ is the disjunction operator.

6. There is a finite set T of portfolios described by the set of criteria $\mathscr{F}$; each portfolio is assigned by the DM each to one category of $\mathbf{C}$. T is called *reference set.*

Our goal is to find a set of parameters $\mathscr{P}' = \{w'_1, \cdots, w'_k, v'_1, \cdots, v'_k, \lambda', \beta'_0\}$ (preference model), that allows one to construct an interval-based outranking model consistent with Equations (4.2.1) to (4.2.6). To achieve this, let us assume that a binary preference relation is built for each $(x, y) \in T \times T$ for a given set of parameters $\mathscr{P}'$ and let us consider the following sets:

$$H_P = \{(x, y) : xP_{\mathscr{P}'}y \text{ with } C_{\mathscr{P}}(x) \not\succ C_{\mathscr{P}}(y)\},$$

$$H_Q = \{(x, y) : xQ_{\mathscr{P}'}y \vee xK_{\mathscr{P}'}y \text{ with } C_{\mathscr{P}}(x) \not\succeq C_{\mathscr{P}}(y)\},$$

$$H_I = \{(x, y) : xI_{\mathscr{P}'}y \text{ with } C_{\mathscr{P}}(x) \neq C_{\mathscr{P}}(y)\}.$$

It is plausible to assume that the cardinalities of these sets must be minimized in order to find the set $\mathscr{P}'$ with the minimum number of inconsistencies. Furthermore, a lexicographical order of importance is evident in the minimization of these cardinalities. Thus, our proposal to find the "best" $\mathscr{P}'$ is to solve the following multiobjective optimization problem:

$$\underset{\mathscr{P}' \in \Gamma}{\text{minimize}} \, (card(H_P), card(H_Q), card(H_I)), \tag{4.2.7}$$

with preferential priority in lexicographical order favoring $card(H_P)$. In (4.2.7), Γ is the set of preference models that fulfill Constraints (2.3.5) and (2.3.6) and $card(\omega)$ is the cardinality of set ω.

4.3 System for approximating the DM's preferences

In order to assess the proposal presented above, we now present a genetic-algorithms-based system that will allow us to obtain approximate solutions to Problem (4.2.7).

Some current research reported in the literature, such as [67], conclude that Genetic Algorithms present more promising results than other meta-heuristics, such as Particle Swarm optimization, Tabu Search and Simulated Annealing, when solving multiobjective optimization problems similar to Problem (4.2.7) but with real numbers. Consequently, in this work we use a Genetic Algorithm capable to deal with parameters defined as interval numbers in order to search for the solution of Problem (4.2.7).

The chromosomes in the proposed Genetic Algorithm consist of the interval based outranking parameters: $w_1, \cdots, w_k, v_1, \cdots, v_k, \lambda, \beta_0$. For example, if we assume that $k = 3$, then the individual is formed as shown in Figure 4.1.

There are $k + 2$ crossing and mutation points. In order to fulfill consistency constraints (2.3.5) and (2.3.6), the weights are all considered as only one gene. The points to perform the crossing and mutation operations in the example are shown in Figure 4.2.

Figure 4.1: Individual representing a solution to Problem (4.2.7), with $k = 3$

w_1	w_2	w_3	v_1	v_2	v_3	λ	β_0

Figure 4.2: $k + 2$ cutoff points for individuals, with $k = 3$

1 2 3 4 5

w_1	w_2	w_3	v_1	v_2	v_3	λ	β_0

Mutation consists of the random generation of a gene. In the previous example, if gene 1 is selected to mutate, then the set of weights would be randomly generated satisfying the consistency constraints (2.3.5) and (2.3.6). If gene 2 is chosen to mutate, then a random value would be generated for the veto of the first criterion, v_1. The probability with which an individual is selected to mutate is $\frac{1}{L}$, where L is the population size. The selection of parents in each generation of the genetic algorithm is done through a binary tournament; the winning individuals in two tournaments are crossed in a single-point crossover to generate an offspring individual.

Algorithm 3 describes the procedure. The algorithm first randomly creates an initial population of L individuals, P_0, fulfilling constraints (2.3.5) and (2.3.6). Each individual's fitness is assessed from objectives in Problem (4.2.7) and based on a reference set χ_m of cardinality m; that is, it calculates the number of (preferentially ordered) inconsistencies where the individual allowed a different binary relation with respect to χ_m. After that, and for each generation, from the current population P_g the algorithm creates an offspring H_g using the chosen operators of Selection, Cross and Mutation, whose fitness is also assessed using Problem (4.2.7). The next step is the combination of parents and descendants in a pool from which the algorithm extracts the individuals with the best fitness. The individuals with the best fitness within the pool form the next generation of parents, P_{g+1}. This procedure is repeated for G generations. Later, the algorithm returns the individual that represents the feasible solutions with the best fitness values in the last population. This vector is obtained as the centroid (average of the parameters) of individuals with the best fitness value. It can be demonstrated that if the centroid is obtained from a set of feasible solutions, then such centroid is also feasible. In order to discard randomness in the procedure, we generate L centroids. And, in order to take advantage of these centroids, we use them as a "seed population" for a last run of the algorithm. The centroid generated in this final run is recommended as the best solution to Problem (4.2.7).

It is possible to incorporate information into the genetic algorithm that will help to reduce the search space. Some ways to add this type of information are the following.

Algorithm 3 Algorithm proposed to address Problem (4.2.7)

Require: L, the population size; G, the number of generations; χ_m, a reference set of cardinality m.

Ensure: ρ_{final}, individual representing the population with the best fitness value.

$i \longleftarrow 1$
for $i \leq L$ **do**
  $g \longleftarrow 0$
  $P_g \longleftarrow$ createInitialPopulation()
  assessFitnessPopulation(P_g, χ_m)
  for $g \leq G$ **do**
    $H_g \longleftarrow$ createOffspring(P_g, selection, crossover, mutation)
    assessFitnessIndividual(H_g, χ_m)
    $P_{g+1} \longleftarrow$ obtainBest($P_g \bigcup H_g$)
    $g \longleftarrow g + 1$
  end for
  $\rho \longleftarrow$ findCentroid(P_{m-1})
end for
$g \longleftarrow 0$
$P_g \longleftarrow \{\rho_1, \rho_2, \cdots, \rho_L\}$
for $g \leq G$ **do**
  $H_g \longleftarrow$ createOffspring(P_g, selection, crossover, mutation)
  $P_{g+1} \longleftarrow$ obtainBest($P_g \bigcup H_g$)
  $g \longleftarrow g + 1$
end for
$\rho_{final} \longleftarrow$ findCentroid(P_m)

- The DM can assign values to some of the parameter boundaries. Because the parameters are expressed as interval numbers, it is relatively easy for the DM to assign the boundaries of some of these parameters. For example, the DM can provide a value μ_j such that if the maximum attainable difference between the performances of portfolios y and x in criterion g_j is equal to or greater than μ_j (that is, if $g_j^-(y) - g_j^+(x) \geq \mu_j$), then there is no doubt that xSy must be vetoed. Therefore, it is possible to limit the search space of the algorithm by doing $v_j^+ = \mu_j$.
- The DM can express that criterion g_j is more important than criterion g_i. In this case, the algorithm must ensure $w_j^- > w_i^+$.
- It must be satisfied that $\lambda^- > 0.5$ and $\lambda^+ < 1$.
- It must also be satisfied that $\beta_0 > 0.5$.

4.4 Validating our proposal to approximate the DM's preferences

This section details the experiments carried out to assess the proposal introduced in Section 4.2. The assessment basically consists in validating the parameters generated by the system described in Section 4.3 in its ability to reproduce the DM's preferences.

4.4.1 Creating experimental instances

To assess the proposal, it is necessary to simulate the DM's decision policy, and to generate sets of instances as case studies. For this purpose, we simulate the DM's preferences through the random generation of the parameter vector $\mathscr{P}$. Each instance of the experiments consists of a reference set containing a finite number of portfolios assigned to categories. Each portfolio is assigned to one of three categories: C_3 = *Good*, C_2 = *Doubt* and C_1 = *Bad*. If a portfolio cannot be assigned to one of the categories consistently with Equations (4.2.1) to (4.2.6), then the current portfolio is rejected and a new portfolio is generated. This procedure continues until the cardinality of the reference set is satisfied.

The first step to create a reference set is to determine a central profile for each category. The central profiles of the categories are assigned in the following way. First, we randomly create a sufficiently large set of portfolios described by the criteria in $\mathscr{F}$. (Sets with 2000 portfolios are used in the experiments described below.) Later, these portfolios are ranked through the outranking net flow score[1] using the

[1]The net flow score is a very popular measure to rank a set of decision alternatives [105]. If $\beta(x, y)$

simulated parameter vector $\mathscr{P}$. Finally, the central profile of a given class is defined as the action with the most representative position within the whole rank. For example, the central profile of the lowest class (C_1) is close to the position $\lceil 2000/6 \rceil$. As stated above, Equations (4.2.1) to (4.2.6) must always be fulfilled when assigning a portfolio to a category.

To assign the rest of portfolios to the categories, we follow the next procedure: i) randomly create a new portfolio described by its impact in the criteria; ii) determine if it can be assigned to a category (fulfilling Equations (4.2.1) to (4.2.6) and using $\mathscr{P}$ as the DM's system of preferences); iii) if it cannot be assigned to any category, go to step i; iv) if it can be assigned to just one category, assign the solution to that category; v) if it can be assigned to more than one category, assign the portfolio to the central category among those categories where the solution fulfills Equations (4.2.1) to (4.2.6).

The bounds of $g_i(x) = [g_i^-(x), g_i^+(x)]$ are generated as $g_i^-(x) = \min\{d_1, d_2\}$, $g_i^+(x) = \max\{d_1, d_2\}$ where $d_j = \min\{10, \max\{1, \hat{i}(1 - \epsilon_j)\}\}$, $\hat{i} \in [1, 10]$, $\epsilon_j \in [-0.3, 0.3]$, $j = 1, 2$, $\hat{i}$ and ϵ_j are randomly generated. The parameters of $\mathscr{P}$ are generated as follows. First, β_0 is randomly generated in (0.5, 0.6) while the bounds of the i-th veto are defined as $v_i^- = 0.7(\hat{v}_i)$ and $v_i^+ = 1.3(\hat{v}_i)$ where $\hat{v}_i$ is randomly generated in [3,5]. We calculate the core values of the weights as $\hat{w} = \frac{1}{n}$ and the weight of criterion g_i as $[w_i^- = (1 - \omega_i)\hat{w}$, $w_i^+ = (1 + \omega_i)\hat{w}]$, where ω_i is randomly generated in [0,0.3]. The lower bound of λ, λ^-, is randomly generated in [0.51,0.76] and its upper bound is set as $\lambda^+ = 1.3\lambda^-$. For all cases $i = 1, \cdots, k$.

During the optimization, these parameters are randomly generated in a similar way, but with wider ranges of search: β_0 and the bounds of λ are randomly generated in (0.5, 1), ω_i is randomly generated in [0.1,0.5], and the bounds of v_i are $v_i^- = 0.5\hat{v}_i$ and $v_i^+ = 1.5\hat{v}_i$.

4.4.2 Validation procedure

We consider that it is important to validate the effectiveness of our proposal when we deal with

- different decision models (e.g., different DMs);
- different number of criteria;
- different number of portfolios in the reference set;
- out-of-sample situations: testing the model's capacity of generalization when approaching new decisions on portfolios out of the reference set.

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is a fuzzy preference relation on a set $\mathscr{A}'$, the net flow score associated to $a \in \mathscr{A}'$ is defined as $F_n(a) = \sum_{c \in \mathscr{A}' - \{a\}} [\beta(a, c) - \beta(c, a)]$ [98].

We are particularly interested in testing the ability of our proposal to create a decision model that reproduces the same preference relations and/or the same assignments as the DM. Therefore, we validate this ability in the context of preference relations and in the context of ordinal classification. We use the following validation procedure.

1. Use two sets of criteria with different cardinality to describe portfolios; namely, six and twelve criteria.
2. Simulate the decision model of a DM through the random generation of the parameter vector $\mathscr{P}$.
3. Create five reference sets, $\chi_{10}, \chi_{20}, \chi_{30}, \chi_{40}$, and χ_{50}, with cardinalities of 10, 20, 30, 40 and 50, respectively, using three categories, $C_3 = Good$, $C_2 = Doubt$ and $C_1 = Bad$. We denote the assignment of portfolio x to category j in the m-th reference set by the decision model $\mathscr{P}$ as $C_{\mathscr{P}_m}(x) = j$.
4. Obtain, through the system described in Subsection 4.3, a set of parameters $\mathscr{P}'$ as consistent as possible with the assignments made by the simulated DM (whose real decision model is $\mathscr{P}$) in each reference set. The *maxima* consistency is identified with the best compromise of Problem (4.2.7) and the optimization is performed using Algorithm 3.
5. Obtain the *in-sample effectiveness* of the proposal when evaluated in the context of *preference relations* as follows: first, determine the preference relation for each pair of portfolios $(x, y) \in \chi_m \times \chi_m$, $m \in \{10, 20, 30, 40, 50\}$ through $\mathscr{P}'$, and call it $xR'_m y$, $R \in \{P, Q, K, I\} \bigcup \{O\}$. $\{P, Q, K, I\}$ is the set of preference relations described in bullet point 4 of Section 4.2, while O indicates that the relation between x and y is not in this set. It is necessary to note here that the definition of the preference relations does not guarantee that one of the four relations will occur between x and y for $\mathscr{P}'$. However, the way that the reference sets are created (Subsection 4.4.1) allows one of these relations to be always hold between each pair of portfolios for $\mathscr{P}$. Thus, the situation where we cannot set one of these relations between each $(x, y) \in \chi_m \times \chi_m$ for $\mathscr{P}'$ shall be counted as an inconsistency between $\mathscr{P}'$ and $\mathscr{P}$.
   Finally, contrast the preference relation obtained through $\mathscr{P}'$ with the one inferred from the assignments made by the DM of x and y to their respective categories in χ_m and calculate an error indicator for the method in each instance of reference set χ_m as
   $$\xi_m = \sum_{(x,y) \in \chi_m \times \chi_m} [\xi_{P_m}(x, y) + \xi_{Q_m}(x, y) + \xi_{K_m}(x, y) + \xi_{I_m}(x, y) + \xi_{O_m}(x, y)] \tag{4.4.8}$$
   where
   $\xi_{P_m}(x, y) = 1$ if $xP'_m y \Longrightarrow C_{\mathscr{P}_m}(x) > C_{\mathscr{P}_m}(y)$ is false and 0 otherwise,
   $\xi_{Q_m}(x, y) = 1$ if $xQ'_m y \Longrightarrow C_{\mathscr{P}_m}(x) \geq C_{\mathscr{P}_m}(y)$ is false and 0 otherwise,
   $\xi_{K_m}(x, y) = 1$ if $xK'_m y \Longrightarrow C_{\mathscr{P}_m}(x) \geq C_{\mathscr{P}_m}(y)$ is false and 0 otherwise,
   $\xi_{I_m}(x, y) = 1$ if $xI'_m y \Longrightarrow C_{\mathscr{P}_m}(x) = C_{\mathscr{P}_m}(y)$ is false and 0 otherwise,
   $\xi_{O_m}(x, y) = 1$ if xOy.

Hence, the effectiveness of the method when evaluated in the context of *preference relations* is defined as

$$1 - \frac{\xi_m}{\eta} \tag{4.4.9}$$

(i.e., the strict negation of the proportion of errors with respect to the number of preference relations); where $\eta = m/2(m-1)$, $m \in 10, 20, 30, 40, 50$.

6. Obtain the *out-of-sample* effectiveness of the proposal as follows: first, use $\mathscr{P}$ to assign new portfolios (different to the ones in the reference sets) as described in Subsection 4.4.1 (here we generate sets of 100 portfolios). Later, use $\mathscr{P}'$ to find the binary preference relations between pairs of these new portfolios. Finally, determine the out-of-sample effectiveness of the proposal when evaluated in the context of preference relations using an analogous validation method as the one used in step 5.

7. Obtain the *out-of-sample* effectiveness of the method when evaluated in the context of *ordinal classification* as the proportion of coincidences between the assignments made by the DM and the one made by the method through $\mathscr{P}'$ and using also the method described in Subsection 4.4.1, both using the portfolios created in step 6.

One instance of the experiment consists in a simulated DM, a given cardinality of the criteria set, and a cardinality of the reference set. We consider 40 instances to be sufficient to perform a satisfactory validation of the method using the validation process described in steeps 4 to 7.

4.4.3 Results

We use the system described above to assess our proposal in its ability finding a decision model such that i) the interval-based outranking method can state the same binary preference relations as the ones inferred from the assignments made by the simulated DM; and ii) a multicriteria methods suggest the same assignments to pre-defined classes as the ones made by the simulated DM.

Validating our proposal in the context of preference relations

Here, we analyze the in-sample and out-of-sample effectiveness of our proposal to state the same binary relation as the ones inferred from the assignments made by the simulated DM. Such analysis is performed both in-sample and out-of-sample with respect to the actions in the reference sets. We evaluate the results obtained when the portfolios are described by six and twelve criteria.

Actions described by six criteria

In-sample effectiveness Table 4.3 shows the average effectiveness of our proposal, as calculated by Equation (4.4.9), and its standard deviation for each reference set. We calculated these results from the effectiveness of our proposal in the 40 instances of the experiment. The Wilcoxon Signed-Ranks test for two paired samples indicated that the difference of each pair of average performances is considered to be statistically significant with a 0.95 confidence level. This means that the increments in the cardinality of the reference sets allowed the model to increase its performance and shows the number of portfolios that the DM should classify in order to obtain an expected performance.

Table 4.3: Average in-sample effectiveness of the proposal relative to preference relations for each reference set using six criteria

Reference set	Average effectiveness	Standard deviation
χ_{10}	0.9822	0.0023
χ_{20}	0.9992	$1.36E^{-05}$
χ_{30}	0.9993	$6.02E^{-06}$
χ_{40}	0.9996	$2.21E^{-06}$
χ_{50}	0.9997	$8.48E^{-07}$

Evidently, an effectiveness lower than 100% is due to the error indicator, χ_m, being greater than zero. Each of the error types, χ_{P_m}, χ_{Q_m}, χ_{I_m}, and χ_{O_m}, has a different level of proportion in the total error (step 5 of Section 4.4.2). Table 4.4 shows the average ratio in χ_m of each of the error types.

Table 4.4: Average proportion of each type of error in χ_m

Reference set	$\xi_{P_m}(x, y)$	$\xi_{Q_m}(x, y)$	$\xi_{K_m}(x, y)$	$\xi_{I_m}(x, y)$	$\xi_{O_m}(x, y)$
χ_{10}	0.43	0	0	0.14	0.43
χ_{20}	0.66	0	0.03	0.03	0.28
χ_{30}	0.64	0	0	0.16	0.20
χ_{40}	0.61	0.01	0.03	0.07	0.28
χ_{50}	0.59	0	0.01	0.12	0.28

Table 4.4 indicates that inconsistencies respect to the strict preference provide most of the error encompassed in the global error. Of course, not all preference relations occur with the same frequency in the experiment. Actually, it is the strict preference the one with the highest frequency, so it is not surprising

that χ_{P_m} has the largest proportion in the error indicator. Table 3 shows the effectiveness with respect to the frequency of each preference relation. This effectiveness is obtained as the average number of times that the preference relation inferred from the simulated DM's assignments and the preference relation found through $\mathscr{P}'$ coincide. For example, with cardinality equal to 50, there are 819 pairs on average (in the 40 instances) where strict preference exists, according to the $\mathscr{P}'$ model. Nevertheless, the χ_{P_m} indicator was only 2 on average (that is, there were only two times where $xP'_{50}y \Rightarrow C_{P_{50}}(x) > C_{P_{50}}(y)$ was false, out of the 819 opportunities where it could happen). Thus, for the case of strict preference, the model has an effectiveness of 99.76% in χ_{50}.

Table 4.5: Average effectiveness by preference relation

Reference set	P	Q	K	I
χ_{10}	0.9973	1.0000	0.9999	1.0000
χ_{20}	0.9994	1.0000	0.9999	0.9998
χ_{30}	0.9997	1.0000	1.0000	1.0000
χ_{40}	0.9999	1.0000	1.000	1.0000
χ_{50}	0.9976	1.0000	1.0000	0.9997

Tables A.1 to A.4 of Appendix A show the comparison of the models $\mathscr{P}$ and $\mathscr{P}'$ in reference sets χ_{30}. The parameter values are rather similar, although there are some significant deviations. But the really important feature of the indirect elicited parameters is to allow making decisions consistent with the reference sets, even when this set of parameters is not alike the ones of the simulated DM.

Out-of-sample effectiveness The final goal of the PDA paradigm is to create a decision model consistent with the DM's preferences; so, the decision model should suggest decisions that may be considered appropriate by the DM. In this subsection we assess out-of-sample the effectiveness of our proposal to find such model; that is, we use the decision models built by the proposed approach to determine binary preference relations between actions different to the ones within the reference sets. We analyze the method's capability to "predict" the preference relation between pairs of portfolios described by six criteria such as the DM would have done it, considering a set of 100 assignments other than those in the original reference sets. These assignments are performed by the same simulated DMs for whom the preference models $\mathscr{P}'$ were created. We measure this effectiveness based on the solution generated by our method in each instance per reference set using the same effectiveness measure from the previous section.

Table 4.6 shows the average effectiveness of the proposed method and its standard deviation for each cardinality of the original reference sets used to build $\mathscr{P}'$. Unlike the results obtained in the in-sample test (Table 4.3), here some pairs of average performances are not statistically different; namely, those of reference sets χ_{20} and χ_{30}, and χ_{40} and χ_{50}. Furthermore, when performing the same statistical test (Wilcoxon Signed-Ranks test for two paired samples with 0.95 confidence level) between the respective reference sets of the in-sample and out-of-sample effectiveness, we saw that the high performance of the model was not maintained when the cardinality of the reference sets was increased to 12 criteria.

Table 4.6: Average out-of-sample effectiveness of the proposal relative to preference relations for each reference set using six criteria

Reference set	Average effectiveness	Standard deviation
χ_{10}	0.9270	0.1150
χ_{20}	0.9878	0.0137
χ_{30}	0.9891	0.0151
χ_{40}	0.9946	0.0061
χ_{50}	0.9947	0.0080

Twelve criteria

In-sample effectiveness Here we show the results obtained when carrying out experiments with portfolios described by twelve criteria. Table 4.7 shows the first results. In this table we can see that the effectiveness of the proposed method when it worked with portfolios described by twelve criteria is actually not worse than with portfolios described by six criteria. The procedure to obtain these results is the same as the one stated in Section 4.4.3.

Table 4.7: Average in-sample effectiveness of the proposal relative to preference relations for each reference set using twelve criteria

Reference set	Average effectiveness	Standard deviation
χ_{10}	0.9794	0.0018
χ_{20}	0.9992	$5.04E^{-06}$
χ_{30}	0.9995	$1.95E^{-06}$
χ_{40}	0.9998	$2.15E^{-07}$
χ_{50}	0.9996	$7.84E^{-07}$

Out-of-sample effectiveness We show now the method's capability to predict the preference relation between pairs of portfolios when these are different to the reference sets used by the approach to build $\mathscr{P}'$. Table 4.8 shows the average effectiveness of the indirect elicitation proposal and its standard deviation for each reference set. The portfolios are described by twelve criteria and the procedure to obtain the results is stated in Section 4.4.3.

Table 4.8: Average out-of-sample effectiveness of the proposal relative to preference relations for each reference set using twelve criteria

Reference set	Average effectiveness	Standard deviation
χ_{10}	0.8634	0.1401
χ_{20}	0.9820	0.0184
χ_{30}	0.9882	0.0158
χ_{40}	0.9911	0.0125
χ_{50}	0.9937	0.0106

Validating our proposal in the context of ordinal classification

We show in this section the results of further experiments where the quality of the found decision models to assign the portfolios in preferentially ordered categories is revised. We analyze this quality by comparing the assignments made by the simulated DM with those made using the decision policies found by the proposed approach. The quality is evaluated out-of-sample using the method described in Subsection 4.4.1 (originally used to create the reference sets).

According to the procedure presented in Section 4.4, we have now the decision models created by our method in two sets of experiments. In the first set, the DM expressed her/his preferences on portfolios described by six criteria, while in the second set, the portfolios were described by twelve criteria. Furthermore, in each set of experiments, five reference sets with 10, 20, 30, 40 or 50 portfolios were used. We consider now a new set of 100 portfolios assigned by the DM to the three categories. The assignments in this set are performed by the DM using the method described in Subsection 4.4.1. Later, we use the decision models created by our method in the two sets of experiments to contrast the assignments. The effectiveness is thus defined as the proportion of times that the model reproduced the assignments made by the DM. The simulated DMs and the decision policies found by our approach that are used during the experiments below are the same as the ones used in Subsection 4.4.3 (see also the validation procedure in Subsection 4.4.2).

Out-of-sample effectiveness in the context of ordinal classification We define the out-of-sample effectiveness of the approach in the context of ordinal classification as the proportion of times that the assignments, made using the procedure described in Subsection 4.4.1 and the decision models found by the proposed approach, coincide with those made by the DM in the new set of 100 actions. Table 4.9 shows this effectiveness per reference set for actions described both by six and twelve criteria.

Table 4.9: Average in-sample effectiveness of the proposal relative to ordinal classification for each reference

Reference set	**Average effectiveness**	
	Six criteria	**Twelve criteria**
χ_{10}	0.8938	0.8433
χ_{20}	0.9210	0.9345
χ_{30}	0.9013	0.9003
χ_{40}	0.9430	0.8603
χ_{50}	0.9625	0.8495

Chapter 5

A novel approach to select the best portfolio incorporating the preferences of the decision maker

This chapter presents our approach to allocate resources to a set of investment objects. The approach models the investor's preferences in order to find the most satisfactory portfolio from his/her perspective when many objective functions are considered. The estimations of the portfolios' future returns, the risk of not attaining those returns and the investor's behavior to this risk are all represented through probabilistic confidence intervals (Chapter 3). The imperfect knowledge related to the subjectivity of the investor is modeled on the basis of Interval Theory (Subsection 2.2) and the interval-based outranking method (Subsection 2.3.4), allowing us to obtain an approximation to the investor's preferences even when these are ill-determined, imprecise, uncertain or arbitrary (even missing, Chapter 4). These preferences are used by the proposed approach to perform an aggregation of all the criteria. Thus, a selective pressure towards the investor's most preferred portfolios is produced, while the investor's cognitive effort in the final selection is reduced.

An illustrative example in the context of stock portfolio optimization is provided, where several investors interested in many criteria are simulated and historical real data is used. The criteria are related to the confidence intervals around the portfolios' returns, and indicators from the so-called fundamental and technical analyses. The performance of our approach is compared with that of an outstanding multiobjective evolutionary algorithm, MOEA/D, and some well-known benchmarks in Modern Portfolio Theory and Finance Theory, namely, the Mean-Variance approach and the Dow Jones Industrial Average index. The results show an evident superiority of the proposed approach in both the context of the underlying criteria (confidence intervals and financial indicators) and the context of the actual returns. Moreover, the DM's satisfaction and the effectiveness in reproducing the DM's decisions using the indirect elicitation of preferences described in Chapter 4 are evaluated and compared with those of a direct elicitation procedure. The results show an evident superiority of the former approach. Thus, we conclude that the proposed approach was able to find satisfactory portfolios in the context of the experiments. Finally, some recommendations about the criteria used in the illustrative example are provided.

5.1 Introduction

In modern society, many objectives are commonly contemplated when generating portfolios (see [15,270, 309,310]). Some of the objectives most commonly mentioned in the related literature are (Subsection 2.1.2):

- Maximization of the portfolio's return (e.g., [195,264]).
- Maximization of social responsibility and ethical considerations (e.g.., [123,128,279]).
- Maximization of liquidity (e.g., [9,158]).
- Maximization of return with respect to some benchmarks (see [270]).
- Maximization of the amount invested in R&D (see [270]).
- Minimization of transaction costs (e.g., [193,297]).

From all these objectives, the most outstanding one is maximization of the portfolio's return/profit [264]. This is sometimes the only objective optimized during the allocation of resources; however, given the high complexity involved in the return's forecasting procedure, many criteria (e.g., expected return, risk, so-called fundamental and technical analyses) usually underlie such objective. Here, we will address, without loss of generality, the latter situation.

Investors frequently use decision-aiding tools in order to obtain a set of portfolios representing, to a certain extent, the best feasible allocations of resources. But this does not solve the problem; the investor still must choose from among all these portfolios the one that represents the best compromise among the considered criteria. But, as reported by Miller [207], this is not a trivial task since the cognitive limitations make it very difficult for the investor to consistently select the best compromise in the presence of many criteria. This becomes more complicated when she/he needs to make trade-offs between risk and return, and when the criteria values are defined as interval numbers. Consequently, a more convenient approach must be followed; the goal of such an approach must be to provide a minimal set of portfolios satisfying the investor's preferences. The main objective of this work is thus to propose an approach able to deal with many criteria in order to create a portfolio that satisfies the preferences of the investor. That is, our approach is intended to find the most preferred portfolio.

Since the groundbreaking work of Markowitz [195], many authors have presented interesting methods to create portfolios with the objective of maximizing the portfolio return (see e.g., [88,126,152,162]). Many types of criteria beyond Probability Theory are usually considered by real investors (decision makers, DMs). These criteria range, for example, from financial information of the investment objects to the behavior of their returns through time. There are two main perspectives that consider these criteria in the context of stock portfolios: one, the so-called fundamental analysis, mainly uses ratios to express the real (and probably hidden) value of the companies underlying the stocks (see e.g., Ref. [287]); whereas the other, the so-called technical analysis, principally uses signals that indicate the goodness of time to execute stock transactions by analyzing their serial returns over time (see e.g., Ref. [19]). These two types of indicators together with the approximation to the probability distribution of the returns constitute the most mentioned criteria in the literature of stock portfolio optimization. The necessity of considering all these aspects during the allocation of resources come from the high volatility of the stocks' returns and the complexity to estimate them. Furthermore, sometimes there is just a too short performance history of some stocks as to obtain a reliable approximation to their probability distribution, and/or insufficiency of available financial information. Hence, addressing a many-criteria optimization problem (i.e., an optimization problem with more than three criteria) describing all these perspectives can be required. On the other hand, the allocation of resources requires the investor to analyze several solution alternatives in order to find the most preferred one. Thus, the decision maker's preferences must necessarily be contemplated. A viable option is to incorporate these preferences in the portfolio optimization process. However, the formulation of all these many aspects in a multicriteria problem to select the best portfolio according to the investor's perspective is scarce. We believe that the lack of popularity of such an approach is mainly due to its high computational cost, caused by an overwhelming number of points in the optimization's search space. Indeed, considering many (more than three) criteria makes the search space grow to a size that makes the Portfolio Optimization Problem not solvable by exhaustive methods. Moreover, the number of alternative solutions in the Pareto front for such problems tend to be also overwhelming, making it very hard for the investor to reach a final decision about what he/she considers the best portfolio.

In order to find the best portfolio, the solution that offers the best compromise among the criteria must be found using the decision maker's particular system of preferences (decision policy). That is, since all the solutions within the Pareto front are mathematically equivalent, the DM should provide additional information for choosing the most preferred one (cf. [129]). This implies that it is necessary to consider the subjectivity of the investor in aspects such as her/his attitude facing risk, the importance that she/he assesses to each criterion, and certain thresholds that dictate when the investor considers that a portfolio is at least as good as another. If these aspects are considered during the search process, a privileged zone within the Pareto front called *region of interest* can be found (cf. [142]). Nonetheless, incorporating the DM's subjectivity could be a hard task, mainly due to: i) the imperfect knowledge about the true values of the parameters representing the investor's subjectivity, and ii) the cognitive effort required from the DM to reduce this source of imprecision. This implies hardness to guarantee that the preferential information directly provided by the DM precisely coincides with his/her actual decisions. Some authors (e.g., [216]) argue that the only valid information that the decision maker can provide about his/her own preferences is in the form of concrete decisions. Furthermore, the direct elicitation process presents several important problematics. For example, more often than not, the access/availability/constancy of investors to engage in a complete explanation of their implicit preferences is limited. It is more common for investors to be willing to participate just in the first step of the optimization process and they tend to feel more comfortable expressing decisions than explaining them. These situations must necessarily be taken into account when modeling preferences in decision aiding [246]. However, to the best of our knowledge, there are not published papers that deal with this type of difficulty when the subjectivity of the investor is incorporated in the Portfolio Optimization Problem considering many criteria. Here, we intend to fill this gap through an approach that allows the investor to consider as many criteria as she/he requires, considers the risk of not attaining the forecasted impacts on these criteria, incorporates the investor's behaviour facing this risk and takes into account her/his preferences in the search process.

In this work, we assume that the situation where the DM is not willing/capable to provide preference information in an interactive way holds and propose an a priori approach based on the outranking method. Unlike other ways of modeling preferences, the outranking method is able to deal with i) ordinal and qualitative information, ii) zones of uncertainty in the investor's mind, iii) intransitive preferences, iv) non-compensatory effects and veto situations, and v) incomparability between solutions. The main argument against the outranking method is its requirement for many preference parameters and the difficulty of eliciting them. Thus, we use here the recent generalization of the outranking method proposed in Ref. [97] (Subsection 2.3.4) that defines the preference parameters as ranges of values instead of defining them as punctual values. So, the DM is now capable to directly provide the parameters values that are most representative of his/her preferences and a well-suited portfolio might be found according to the DM's decision policy. Below we provide evidence that, even when this new ability of the outranking method makes it easier to directly elicit the DM's preference values, the indirect elicitation proposed in Chapter 4 allows to obtain statistically better (more preferred) solutions.

An illustrative example in the context of the stock Portfolio Optimization Problem is provided. The dataset used in the validation consists in the actual monthly returns of the stocks within the Dow Jones Industrial Average (DJIA, Subsection 2.1.4.3) index during the period April 2011-March 2016. The proposed approach is evaluated in both the context of the original criteria and in the context of the actual returns. With respect to the former, we demonstrate that the portfolios constructed by our approach are satisfactory from the investor's perspective. With respect to the latter, comparisons with the actual returns of the DJIA index and other benchmarks show that the performance of the proposed approach is evidently better.

The chapter is structured as follows. Our proposal is detailed in Section 5.2. Section 5.3 provides the validation process and shows the results obtained. Within Section 5.3, we first formalize the problem that the proposed approach will address, later we describe the experimental design followed to assess our approach's effectiveness and, finally, the results of this assessment are provided.

5.2 A bi-criteria formulation based on Fuzzy Logic

In order to formalize our proposal, we first assume that the DM's implicit system of preferences can be represented by the set of parameters $\mathscr{P} = \{w_1, \cdots, w_k, v_1, \cdots, v_k, \lambda, \beta_0\}$. This set of parameters allows us to build an interval-based outranking relation between pairs of portfolios. Then, we take advantage of this method's capacity to estimate the likelihood of the following two statements: i) "portfolio x is at least as good as portfolio y" and ii) "portfolio x dominates portfolio y" (see Subsection 2.3.4). With such values, we then obtain a *non-outranked likelihood degree* to estimate how "preferred" a portfolio is with respect to a given set of portfolios.

In the context of real-valued parameters, Fernandez et al. [95,98] define a portfolio x to be strictly-non-outranked if and only if there is not a portfolio y such that y dominates x, or if y outranks x and x does not outrank y. Formally:

x is non-outranked $\iff \neg\exists y\ :\ yDx \vee (ySx \wedge \neg xSy)$

$$\equiv \forall y\ :\ \neg yDx \wedge (\neg ySx \vee xSy).$$

Using μ to denote "truth degree" and the strict negation operator, we formulate the previous definition in terms of Multivalent Logic as:

$$\mu(x \text{ is strictly non-outranked}) = \mu(\forall y\ :\ ((1 - \mu(yDx)) \wedge ((1 - \mu(ySx)) \vee \mu(xSy)))).$$

Among different logic approaches, we use here the so-called Compensatory Fuzzy Logic [83,86], which has several desirable properties for rational decision-making (Subsection 2.5.4). The compensatory logic operators for conjunction have as limits the minimum operator [300]. Other compensatory logic operators are the arithmetic mean and the geometric mean. The latter is considered as the simplest among the quasi-arithmetic means (cf. [83,86]). Unlike the minimum operator, the geometric mean satisfies the strict growth axiom of the compensatory fuzzy logic [82]. In this work, the conjunction and disjunction operators from Compensatory Fuzzy Logic based on the geometric mean are taken to obtain a non-outranked degree as follows.

Let U be the universe of portfolios within the Pareto front of Problem (1.2.1). For each pair $(x, y) \in \mathscr{A} \times \mathscr{A}$, $\mathscr{A} \subseteq U$, it is possible to obtain through the interval-based outranking approach: i) a likelihood index of the assertion "x outranks y", denoted by $\beta(xSy)$; ii) a likelihood index of the assertion "y outranks x", denoted by $\beta(ySx)$; and iii) a likelihood index of the assertion "x dominates y", denoted by $xD^{\alpha}y$. Now, let us make $\mu(xSy) = \beta(xSy)$, $\mu(ySx) = \beta(ySx)$, $\mu(xDy) = xD^{\alpha}y$, $\hat{A} = \mathscr{A} - \{x\}$ and $n = card(\hat{A})$, then we define the *non-outranking truth degree* of x in $\mathscr{A}$ by means of the compensatory fuzzy logic based on the geometric mean as (cf. [82] and Subsection 2.5.4):

$$NS_{\mathscr{A}}(x) = \sqrt[n]{(NS(x, \hat{A})}.$$

Where

$$NS(x, \hat{A}) = \prod_{y \in \hat{A}} \sqrt{((1 - \mu(yDx))(1 - \sqrt{((1 - (1 - \mu(ySx)))(1 - \mu(xSy)))))}$$

A high non-outranked degree of portfolio x indicates the lack of arguments to believe that there are solutions better than x. On the other hand, a high non-outranked degree is a necessary condition to be the best compromise, but it is not sufficient. A solution may have a high non-outranked degree and be incomparable with all or many of the solutions in the known Pareto front. Positive alternative arguments are required to affirm the superiority of x over the other optimal solutions under consideration.

In order to enhance the preference information, here we suggest to use the outranking net flow score. This is a very popular measure to rank a set of decision alternatives on which a fuzzy preference relation is defined (cf. [104]). If $\beta(xSy)$ is an outranking likelihood index on set $\mathscr{A}$, then the net flow score associated to $x \in \mathscr{A}$ is defined as $F_n(x) = \sum_{y \in \mathscr{A} - \{x\}} (\beta(xSy) - \beta(ySx))$. Note that $F_n(x) > F_n(y)$ is an asymmetric and transitive binary relation on $\mathscr{A}$, indicating to some extent preference of x over y. So, the net flow score may be used to select the most satisfactory solution between x and y when $NS_{\mathscr{A}}(x) = NS_{\mathscr{A}}(y)$. Nonetheless, always that $NS_{\mathscr{A}}(x) > NS_{\mathscr{A}}(y)$, the DM can be confident that portfolio x provides her/him more satisfaction than portfolio y, regardless the values of $F_n(x)$ and $F_n(y)$. Therefore, a best compromise solution can be found through a lexicographic search.

Taking into account the non-outranking truth degree and the net flow score information, we propose to select the best solutions for Problem (1.2.1) as the non-dominated set of solutions obtained from:

$$\underset{x \in \Omega}{\text{maximize}} \{NS_{\mathscr{A}}(x), F_n(x)\} \tag{5.2.1}$$

with preemptive priority favoring $NS_{\mathscr{A}}(x)$, where Ω is the set of feasible portfolios.

5.3 An illustrative example: a highly risk-averse investor interested in many criteria

We present in this section a study case where the proposed approach is used to create portfolios on the basis of many criteria. We assume that the DM is interested in using confidence intervals (Chapter 3), fundamental indicators (Subsection 2.1.4.1), and technical indicators (Subsection 2.1.4.2) as underlying criteria to the maximization of portfolio return. This assumption reflects that there are some scenarios where the DM is not fully satisfied with the information provided by the statistical analysis nor by the financial analyses. Of course, the assumption made in this illustrative example can be adjusted according to the specific context and DM's requirements.

5.3.1 Problem definition

Although the fundamental and technical analyses are widely used by investors in the real world, the combination of both types of analyses is not common in the academic literature (see Section 2.1.4). Even less common is the combination of fundamental analysis, technical analysis and decisions on proportions in which the resources should be allocated. Nevertheless, the statistical information might not be available/reliable, and/or the financial information might not be enough to involve the risk caused by the return's volatility. Hence, the DM would consider valuable to perform a portfolio optimization where all the criteria are combined in a multicriteria optimization problem. To the best of our knowledge, there are no published works in the related literature considering the three analyses in a multicriteria optimization problem that is also capable of representing the DM's attitude facing risk as well as her/his decision policy in the context of Portfolio Optimization Problem with many criteria.

There are papers in which some of these analyses are used consecutively (e.g., [103,287]). Typically, the fundamental analysis is performed first in order to select the stocks that will be in the portfolio; the technical analysis is performed secondly to determine the time convenience of investing in each stock; finally, the portfolio creation analysis is carried out afterwards to define the allocation proportions to be assigned. Thus, the value of each financial indicator is evaluated individually for each stock. Since our purpose is to select the approximation to the best portfolio, our solution alternatives are not individual stocks, but portfolios. Hence, we use here a way of evaluating portfolios through financial indicators; such way is on the basis of Fuzzy Logic. Particularly, Fuzzy Logic is used to define the truth degree of each stock being "good" according to the financial indicators. Let us now describe our proposal to determine a *quality index* of a portfolio on the basis of these analyses.

5.3.1.1 Fundamental analysis as a criterion to evaluate portfolios

In order to determine a comprehensive quality index of a portfolio with respect to the fundamental indicators, we first need to assess each of the stocks within the portfolio. We assume that a stock is good from the fundamental analysis' viewpoint if the following two conditions are fulfilled:

i. In a significant majority of the fundamental indicators considered, the value of each indicator reaches a sufficiently high level.

ii. No indicator has a value significantly lower than certain threshold n_2.

To model the truth degree of condition i, it is only necessary to define what the DM means by "an important majority" (represented by a relative value τ) and "a sufficiently high level" (represented by a relative number n_1)). A piecewise linear function H can be used here, where the independent variable is the proportion D of indicators that reach level n_1, and fulfills: a) $H = 0$ if D is not greater than 0.5, b) H linearly increases to 1 when D grows from 0.5 to τ, and c) H is 1 for values of D not lower than τ. Thus, H can be defined as follows:

$$H = \begin{cases} 0 & D \leq 0.5, \\ (D - 0.5)/(\tau - 0.5) & 0.5 < D < \tau, \\ 1 & D \geq \tau. \end{cases}$$

The truth degree of condition ii can be modeled by a piecewise linear function f_j^i where the value of the jth fundamental indicator when analyzing the ith stock, $value_j^i$, is the independent variable and f_j^i has the following characteristics: a) f_j^i is zero when the value of the indicator is not greater than the level n_{veto}, b) f_j^i linearly increases to 1 when the value of the indicator moves from n_{veto} to n_2, and c) f_j^i is 1 when the value of the indicator is not lower than n_2. Thus, f_j^i can be defined as follows:

$$f_j^i = \begin{cases} 0 & value_j^i \leq n_{veto}, \\ (v - n_{veto})/(n_2 - n_{veto}) & n_{veto} < value_j^i < n_2, \\ 1 & value_j^i \geq n_2. \end{cases}$$

The truth degree that all indicators have value greater than or equal to n_2 when evaluating the ith stock, is obtained by the conjunction of the values of all f_j^i. An evident compensation exists among such values. Hence, we propose to use the conjunction of the Compensatory Fuzzy Logic based on the geometric mean (cf. Refs. [82,83,86] and Subsection 2.5.4).

Finally, the truth degree of the ith stock being good from the fundamental analysis' viewpoint, F_i, is obtained by the conjunction of the truth values of conditions i and ii. There is no compensation in such conjunction. Hence, we propose to use here the product norm as the conjunction operator.

The aggregation to evaluate the portfolio from this viewpoint then becomes:

$$\overline{F}(x) = \sum_{i=1}^{n} F_i x_i.$$

5.3.1.2 Technical analysis as a criterion to evaluate portfolios

The evaluation of individual stocks using the technical analysis described in Section 2.1.4.2 consists in finding the convenience of investing in the stocks. Particularly, if the rule associated to the jth technical indicator states that the ith stock is good, then such indicator takes a value of 1 ($it_j^i = 1$), otherwise its value is 0 ($it_j^i = 0$). Therefore, the aggregation

$$T_j(x) = \frac{\sum_{i=1}^{n} x_i it_j^i}{n},$$

represents the desirable momentum proportion of the stocks supported by portfolio x from the jth technical indicator perspective. A final aggregation of the technical indicators can be performed to obtain the goodness of portfolio x in the technical analysis' viewpoint:

$$\overline{T}(x) = \frac{\sum_{j=1}^{TN} T_j}{TN},$$

where TN is the number of technical indicators considered.

5.3.1.3 Multicriteria optimization problem

Of course, there is some uncertainty involved in the definitions of $\overline{F}(x)$ and $\overline{T}(x)$ originated in the finite-precision arithmetic provided by computers. Hence, we take advantage of Interval Theory and redefine the financial indicators as interval numbers:

$$F(x) = [F(x)^-, F(x)^+]. \tag{5.3.2}$$

Where $F(x)^-$ is $\overline{F(x)}$ rounded down to four digits, and $F(x)^-$ is $\overline{F(x)}$ rounded up to four digits. The same procedure is followed with the technical indicators to create $T(x)$:

$$T(x) = [T(x)^-, T(x)^+]. \tag{5.3.3}$$

Both $F(x)$ and $T(x)$ can be seen as quality indexes indicating the convenience of investing in portfolio x.

On the other hand and accordingly to Subsection 3.4.1, a highly risk-averse DM can be simulated as the one who requires information about two confidence intervals: one containing the expected return with a

70% of probability, $\theta_{\gamma_{70}}(x)$, and another containing it with the 99% of probability, $\theta_{\gamma_{99}}(x)$ (see Chapter 3 for the motivation on using confidence intervals). Here, as in Ref. [264], three constraints are used: budget, non-negativity, and bounds on individual stocks constraints.

Therefore, we assess the approach proposed in this thesis through its solutions' performances when solving the following multicriteria optimization problem on the basis of confidence intervals and financial indicators:

$$\underset{x\in\Omega}{\text{maximize}} \left(\theta_{\gamma_{70}}(x), \theta_{\gamma_{99}}(x), F(x), T(x)\right). \tag{5.3.4}$$

Where Ω is the set of portfolios fulfilling the following constraints:

$\sum x_j = 1 \longrightarrow$ Budget constraint;

$x_j \geq 0 \longrightarrow$ Non-negativity constraints;

$x_j \leq 0.4 \longrightarrow$ Bounds on individual stocks.

And where

x_j is the proportion of resources allocated to the jth stock,

$\theta_{70}(x) = [\alpha_{70}, \beta_{70}] \; : \; P(\alpha_{70} \leq R(x) \leq \beta_{70}) = 0.70,$

$\theta_{99}(x) = [\alpha_{99}, \beta_{99}] \; : \; P(\alpha_{99} \leq R(x) \leq \beta_{99}) = 0.99,$

$R(x)$ is a random variable representing the actual return of portfolio x,

$F(x)$ is the evaluation of portfolio x from the fundamental analysis' viewpoint (Equation (5.3.2)),

$T(x)$ is the evaluation of portfolio x from the technical analysis' viewpoint (Equation (5.3.3)), and

$j = 1, \cdots, n.$

Given the exponential increase in the number of solutions required for approximating the entire Pareto front of Problem (5.3.4), an incorporation of the DM's preferences is desirable [142]. Several authors (e.g., [71,72,142]) argue that it is common for evolutionary multiobjective optimization methodologies to suffer serious difficulties when dealing with four or more criteria. One of these difficulties is the need of a larger number of points to represent a higher-dimensional Pareto optimal front. Such difficulty is worsened when the criteria in the optimization problem are defined as interval numbers. Therefore, finding a preferred and smaller set of Pareto-optimal solutions, instead of the entire front, tends to be beneficial for the search process [72] and it can be achieved by incorporating preference information in the search process [102, 277].

5.3.2 Experimental design

Aiming to provide an extensive assessment of our approach, we present in this subsection an analysis procedure where the two manners to elicit the investor's system of preferences are used, namely the direct and indirect procedures. In order to be able to compare the performance of our approach in both scenarios, we assume that i) the DM's actual preference model is in terms of the interval-based outranking model, and ii) the actual values of the parameters in the preference model are already known. Thus, we first obtain these actual values (through simulation) and later use the elicitation methods to obtain an approximation to such values. Let us now describe the simulation procedure used during the assessment of our approach.

5.3.2.1 Simulation of decision makers

During the experiments, we first generate 20 decision models at random that represent 20 decision makers' decision policies. That is, we create 20 sets of parameters $\mathscr{P}^i = \{w^i_j, v^i_j, \lambda^i, \beta^i_0\}$ $(i = 1, \cdots, 20; j = 1, \cdots, 4)$. The values of the parameters to create each $\mathscr{P}^i$ are uniformly randomly taken from ranges of numbers that work as sources. Such sources are shown in Table 5.1. Recall that β^i_0 is the only real number of the interval-based outranking model's parameters, whereas the rest of parameters are defined as interval numbers. Thus, the sources in rows λ^i, w^i_j and v^i_j are actually used for each of these parameters' bounds. Of course, it is satisfied for the lower bounds of these parameters to be not greater than their respective upper bounds. Particularly, we calculate the weight of criterion g_i as $w^-_i = (1 - \omega_i)\hat{w}$, $w^+_i = (1 + \omega_i)\hat{w}$, where ω_i is randomly generated in [0,0.3] and $\hat{w} = 1/4$. In Table 5.1, the value v^j_{max} is used to represent the maximum impact in the jth criterion of a set of 2000 randomly created portfolios. Constraints (2.3.5) and (2.3.6) settled by the interval-based outranking are also fulfilled in the creation of each $\mathscr{P}^i$.

Table 5.1: Sources used to uniformly randomly assign values to the parameters of the interval-based outranking

Parameter	Source
β^i_0	(0.5,0.6)
λ^i	(0.5,0.6)
w^i_j	(0,1)
v^i_j	$(0.3,0.5)v^j_{max}$

5.3.2.2 A direct elicitation of the DM's system of preferences

In order to address Problem (5.3.4), we use the interval-based outranking approach (Subsection 2.3.4) to model the DM's preferences and to build the aggregation described in Subsection 5.2. Since that approach allows the DM to provide imprecise values for its parameters, it is relatively easy to obtain such values directly from the DM. However, in general terms the elicitation of a preference model's parameters comprises some part of arbitrariness, imprecision, and ill-determination [97]. According to Ref. [97], this is particularly true when "the entity in charge of the decision is a group where its members disagree concerning the parameter values, or when the decision-maker is a mythical or an inaccessible person". Consequently, in the experiments described below we assume that although the values of the interval-based outranking's parameters are directly elicited, there exists some deviation from the most appropriate parameters' values.

Therefore, we randomly deviate each parameter (bound in the case of the interval numbers) in the simulated decision models between 0.1 and 0.3 to obtain 20 new sets $\mathscr{P}^{i\prime} = \{w_j^{i\prime}, v_j^{i\prime}, \lambda^{i\prime}, \beta_0^{i\prime}\}$ $(i = 1, \cdots, 20; j = 1, \cdots, 4)$. These sets simulate the values directly elicited from the DMs and they are used as the actual decision policies in the experiments below.

5.3.2.3 An indirect elicitation of the DM's system of preferences

We present here an experimental design to assess, in the context of the Portfolio Optimization Problem with many criteria, the proposal introduced in Chapter 4. We evaluate both the indirect elicitation method and the genetic-algorithm-based system described there.

Given that this subsection performs an indirect elicitation, a creation of reference sets is now presented (cf. Subsection 4.4.1).

Creating reference sets Each instance i uses a reference set T containing 20 portfolios assigned to categories consistently with the corresponding decision maker's $\mathscr{P}^i$ and constraints (4.2.1) to (4.2.6). Each portfolio is assigned to one of three categories: $C_3 = Good$, $C_2 = Doubt$ and $C_1 = Bad$. The assignments of portfolios to categories are made guaranteeing (as much as possible) a uniform number of portfolios among the categories. If a portfolio cannot be assigned to one of the categories consistently with $\mathscr{P}^i$ and constraints (4.2.1) to (4.2.6), then the portfolio is discarded and a new one is generated. This procedure continues until the cardinality of the reference set is satisfied.

Let us now describe the assignment technique. First, we generate a central profile for each category. To generate the central profiles, we randomly create a sufficiently large set of dummy portfolios described by the family of criteria in $\mathscr{F} = \{g_1, \cdots, g_4\}$. (Sets of 2000 portfolios are used in the experiments described below.) Then, these portfolios are ranked through the outranking net flow score using the corresponding simulated parameter vector $\mathscr{P}^i$. Finally, the central profile of a given category is defined as the portfolio with the most representative position within the whole rank. For example, the central profile of category C_1 is in the position $\lceil 2000/6 \rceil$ if it fulfills constraints (4.2.1) to (4.2.6). If it does not fulfill the constraints, the procedure looks for another portfolio with a close position. To assign the rest of actions within the reference set to the categories, we follow the next procedure: i) randomly create a new action described by its impact in the criteria; ii) determine if it can be assigned to a category (fulfilling constraints (4.2.1) to (4.2.6)); iii) if it cannot be assigned to any category, go to step i; iv) if it can be assigned to just one category, assign the solution to that category; v) if it can be assigned to more than one category, assign the action to the central category (arbitrarily choosing when there is more than one).

For each instance i and in each method of elicitation, it is plausible to assume that the DM has to be satisfied with $\mathscr{P}^{i\prime}$. Thus, we assume that she/he requires i) the importance of the confidence intervals to be greater than the importance assigned to the rest of criteria, since she/he is considered to be highly risk-averse; and ii) the order of importance assigned to the criteria in $\mathscr{P}^i$ must be respected in $\mathscr{P}^{i\prime}$.

5.3.2.4 Portfolio optimization

For each instance i, we obtain the best compromise portfolio in the following way. First, the approximation to the returns' probability distribution in the form of confidence intervals is obtained by Montecarlo simulation; this simulation uses the stocks' historical monthly returns of 36 periods as input and runs 200 statistical points using a pseudo-random numbers generator known as *Mersenne Twister* (see [198,199]). Then, the evaluation of the financial indicators for the portfolio is obtained using Equations (5.3.2) and (5.3.3). Once the portfolios' fitness of a subset of candidate solutions has been achieved with respect to Problem (5.3.4), each portfolio's fitness is aggregated using the DM's parameters, $\mathscr{P}^{i\prime}$, and the procedure described in Subsection 5.2. Finally, the set of best compromise solutions to Problem (5.3.4) is composed with the non-dominated solutions to the underlying Problem (5.2.1).

Since Problem (5.2.1) was raised as a lexicographic non-linear optimization problem, we use here Differential Evolution to address it. Such meta-heuristic generally has good performance in non-linear single objective optimizations (see e.g., [164,165]). Differential Evolution algorithm applied here uses $ps = 100$ individuals as its population size, its stopping criterion is the achievement of $gn = 100$ generations, and it uses an n-dimensional real-valued vector to encode the individuals. Recall that Differential Evolution requires the setting of four additional control parameters (cf. [175]): the crossover probability, CR; the mutation rate, p_m; the differential weight, F; and the distribution index, η. As done in Ref. [175], here we set these parameters as $CR = 1$, $p_m = 1/ps$, $F = 0.5$, and $\eta = 20$. Algorithm 4 presents the pseudo-code of our algorithm.

Algorithm 4 Algorithm proposed to select the most preferred portfolio

Require: DM's preferences (described through the interval-based outranking method; that is, $\mathscr{P} = \{w_j, v_j, \lambda, \beta_0\}$; $i = 1, \cdots, 20; j = 1, \cdots, 4$), Problem context (probability distributions of the stock returns, fundamental indicators for each stock, technical indicators for each stock).

Ensure: Set of portfolios recommended by our approach as the best portfolios, ρ_{best}.

$i \longleftarrow 1$
for $i \leq ps$ **do**
    $g \longleftarrow 0$
    $P_g \longleftarrow$ CreateInitialPopulation()
    for $g \leq gn$ **do**
        $j \longleftarrow 1$
        for $j \leq ps$ **do**
            $H_g^j \longleftarrow$ CreateOffspring(P_g, selection, crossover, mutation)
            $P_g^j \longleftarrow$ SelectBestIndividual(P_g^j, H_g^j)
            $j \longleftarrow j + 1$
        end for
        $g \longleftarrow g + 1$
    end for
    $\rho_i \longleftarrow$ SelectBestFromSet(P_g)
    $i \longleftarrow i + 1$
end for
$g \longleftarrow 0$
$P_g \longleftarrow \{\rho_1, \rho_2, \cdots, \rho_{ps}\}$
for $g \leq gn$ **do**
    $j \longleftarrow 1$
    for $j \leq ps$ **do**
        $H_g^j \longleftarrow$ CreateOffspring(P_g, selection, crossover, mutation)
        $P_g^j \longleftarrow$ SelectBestIndividual(P_g^j, H_g^j)
        $j \longleftarrow j + 1$
    end for
    $g \longleftarrow g + 1$
end for
$\rho_{best} \longleftarrow$ SelectBestFromSet(P_g)

Algorithm 4 first creates an initial population by randomly sampling from Ω. Always that we create a feasible individual, we calculate its fitness in the sense of Problem (5.3.4). That is, we estimate its confidence intervals around the individual's return and its quality indexes according to the financial analyses. Then, for each generation j and for each individual P_g^j of P_g, the algorithm creates a feasible individual H_g^j by applying the Selection, Crossover and Mutation operators of Differential Evolution. After that, P_g^j and H_g^j are compared on the basis of Problem (5.2.1). The best individual is now P_g^j. These steps constitute one generation; we perform $gn = 100$ generations. After performing this number of generations, we obtain the set of individuals ρ_i (likely with cardinality of one) within P_g whose fitness is the best in the sense of Problem (5.2.1). All the previous is considered as the ith run. Several runs (up to ps) are performed to obtain a "seed population" of size ps whose individuals are the solutions found in the previous runs. A final run is performed using the seed population as the initial population. The set of best compromise solutions (likely with cardinality of one) to Problem (5.2.1) in this final run is presented to the DM as the best portfolios.

The main difference between Algorithm 4 and other approaches is the exploitation of the non-outranked truth degree, NS_x, to select the best portfolio(s) according to the decision maker's preferences. NS_x is used as the representative value that reflects the overall satisfaction of x with respect to a set of portfolios according to the decision maker supplied preference information. Such preference information is in terms of the interval-based outranking method; that is, we consider the weights assigned by the decision maker to each criterion, his/her veto values and his/her thresholds about when a solution is at least as good as another.

5.3.2.5 Dataset

We use the historical monthly returns of the stocks in the DJIA index for the period April 2011-March 2016 to perform a back-testing strategy (cf. [220]); the evaluation of our approach is in the period April 2014-March 2016 (3 years of data are used as the training period for the statistical simulation of returns). That time span is recent, it has several upward, downward, and horizontal market's movements, so it is interesting to analyze it. We use here a sliding time window of 36 months/1 month, similarly to Refs. [124,181,264] to perform the back-testing. That is, we use three years for training the statistical model (e.g., we obtain metrics of the data set April 2011 to March 2014) and one month for validation (we will use the metrics obtained to create a portfolio in April 2014). The process is then repeated for each period of one month (in a sliding window manner) until the end of the evaluation period. In other words, we select the best stock portfolio of the current month by using the historical metrics of the previous three years, addressing Problem (5.3.4), and maintaining the portfolio over a one-month investment horizon. Each time we start a new investment horizon, we review the stock portfolio (i.e., we select a new distribution of resources among the stocks) according to the corresponding horizon's valuation.

As done by many authors (e.g., [10,56,124,187,302]), the historical prices used to estimate the returns' probability distribution and to calculate the technical indicators, as well as the returns of the index are obtained from Ref. [289]. And, as done by other works (e.g., [91,212]), the financial data to calculate the fundamental ratios is obtained from Ref. [215]. All the data used, together with all the results obtained, are available for consultation upon request.

5.3.3 Results

The experiments described above allow us to obtain

- The performance of our approach with respect to comparisons with benchmarks in two contexts: the criteria space and the objective space. In the former, we make a comparison in terms of the criteria considered in Problem (5.3.4) and estimate the DM's satisfaction by means of the outranking relation. In the latter, the comparison is in terms of the actual returns of the solutions and a more "practical" point of view about our approach's performance is provided.
- Our approach's ability to reproduce the DM's decisions depending on the model

used to elicit the values of the preference parameters; thus, providing an idea of which method should our approach's practitioner implement.

5.3.3.1 Assessing the performance of our approach

Performance in the criteria space Here, we assess the performance of the portfolios built by our approach using both the direct and indirect elicitation methods. We refer to the former as Direct Elicitation Selective Pressure (D_{sp}) approach, and to the latter as Indirect Elicitation Selective Pressure (I_{sp}) approach. In order to make a clearer exposition of the results obtained, we first show a comparison of the D_{sp} approach with a benchmark and, after concluding an evident superiority of the former, we contrast its results with the results obtained by the I_{sp} approach.

***D_{sp}* vs MOEA/D** The benchmark used as reference to assess the D_{sp} approach is MOEA/D, a state-of-the-art multiobjective evolutionary algorithm based on decomposition [175,295] (Subsection 2.4.2). The goal of such comparison is to provide a reference to the capacity of the approach proposed in Subsection 5.2 to deal with many criteria and obtain satisfactory solutions.

During the exploitation process of MOEA/D, the individuals are represented as real vectors and three randomly selected individuals are used for the crossover operator. The crossover operator works as follows. Let qG_1, qG_2, qG_3 be the quantity of genes satisfying $x_i > 0$ in parent 1, parent 2 and parent 3, respectively. The idea is that the parents provide similar proportions of genetic material to the offspring. So, the number of genes satisfying $x_i > 0$ in the child solution is up to $qG_C = \frac{qG_1+qG_2+qG_3}{3}$ and each parent gives $\frac{qG_C}{3}$ randomly chosen genes to the offspring solution. The mutation operator simply consists in swapping two randomly chosen genes of the offspring solution. The probability of mutation is $p_m = \frac{1}{ps}$, where ps is the population size. In preliminary experiments, we found that discarding the infeasible solutions is the most suitable method to obtain solutions with good performance. The Tchebycheff method (Equation (2.4.7)) is used to aggregate the criteria (cf. [295] and Subsection 2.4.2). The dataset described in Subsection 5.3.2.5 and the constraints defined in Problem (5.3.4) are used here to create the benchmark portfolios.

Both D_{sp} and MOEA/D achieve good approximations to the Pareto front with respect to each other, since they produce a high number of non-dominated solutions (cf. Subsection 2.3.4 to see the definition of interval-based dominance). From the approximations achieved by the approaches in each of the 24 periods of the dataset and for each instance, on average, roughly 0.7% of the solutions found by MOEA/D are dominated by at least one of our solutions, and roughly 0.3% of our solutions are dominated by at least one of the solutions from MOEA/D. Indicating that, if MOEA/D is achieving a good approximation to the true Pareto front of Problem (5.3.4), then our proposal is also obtaining a good approximation.

It is now interesting to know how "good" the constructed portfolios are from the DM's perspective. Thus, we use the DMs' actual systems of preferences, $\mathscr{P}^i$ ($i = 1, \cdots, 20$), (as opposed to the one used by the D_{sp} approach, $\mathscr{P}^{i'}$, Subsection 5.3.2.4) to compare the solutions built by both approaches. Given that each $\mathscr{P}^i$ already has all the parameters needed by the interval-based outranking approach, the comparison of solutions is on the basis of such method. Our intention is to find the proportion of times that the strict outranking relation is met between the portfolios created by each approach. Just to provide an example, Table 5.2 shows a portfolio (arbitrarily chosen) from MOEA/D's approximation to the Pareto front and a portfolio built by the D_{sp} approach for one (also arbitrarily chosen) DM in the April 2014 period. As it was specified above, the unit used in Table 5.2 is the proportion of money to invest in each of the n investment objects. Table 5.3 shows their respective impact on criteria of Problem (5.3.4), and Table 5.4 shows the chosen DM's system of preferences.

Table 5.2: Arbitrarily chosen portfolios built in the period April 2014 using MOEA/D and our approach using a direct elicitation of the DM's system of preferences, D_{sp}

Stock	MOEA/D	D_{sp}
American Express Company (AXP)	0	0
Boeing Co. (BA)	0	0

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Table 5.2 – *Continued from previous page*

Stock	**MOEA/D**	D_{sp}
Caterpillar Inc. (CAT)	0	0
Cisco Systems, Inc. (CSCO)	0	0
Chevron Corporation (CVX)	0	0
EI du Pont de Nemours and Co (DWDP)	0	0
Walt Disney Company (DIS)	0	0
General Electric Company (GE)	0	0
Goldman Sachs Group Inc. (GS)	0	0
Home Depot, Inc. (HD)	0.08	0
International Business Machines Corporation (IBM)	0	0.256
Intel Corporation (INTC)	0.34	0.03
Johnson and Johnson (JNJ)	0	0
JPMorgan Chase and Co. (JPM)	0	0
Coca-Cola Company (KO)	0.24	0
McDonald's Corporation (MCD)	0.03	0.349
3M Co. (MMM)	0	0
Merck and Co., Inc. (MRK)	0.30	0
Microsoft Corporation (MSFT)	0	0
Nike Inc. (NKE)	0	0
Pfizer Inc. (PFE)	0	0
Procter and Gamble Co. (PG)	0	0
ATandT Inc. (T)	0	0.365
Travelers Companies Inc. (TRV)	0	0
UnitedHealth Group Inc. (UNH)	0	0
United Technologies Corporation (UTX)	0	0
Visa Inc. (V)	0	0
Verizon Communications Inc. (VZ)	0	0
Wal-Mart Stores Inc. (WMT)	0	0
Exxon Mobil Corporation (XOM)	0	0

Table 5.3: Evaluation in the criteria space of the portfolios built by the approaches

Criterion	MOEA/D	D_{sp}
70 percent confidence interval	[-0.0108, 0.0293]	[-0.0046, 0.0270]
99 percent confidence interval	[-0.0372,0.0469]	[-0.1747, 0.0452]
Fundamental analysis' quality index	[0.3537,0.3538]	[0.8268,0.8269]
Technical analysis' quality index	[0.3414,0.3415]	[0.2244,0.2245]

Table 5.4: System of preferences of the arbitrarily chosen DM

β_0	w_1	w_2	w_3	w_4
0.55	[0.18,0.24]	[0.56,0.78]	[0.05,0.07]	[0.05,0.06]
λ	v_1	v_2	v_3	v_4
[0.51,0.56]	[0.03,0.04]	[0.10,0.12]	[0.27,0.32]	[0.10,0.13]

Now, let y and x be the portfolios shown in Table 5.2, which were built using MOEA/D and D_{sp}, respectively. According to the interval-based outranking method, the likelihood indexes that the DM (in Table 5.4) assigned to these portfolios are shown in Table 5.5. From this table we can deduce that, although there is no dominance between the portfolios, the DM is more satisfied with the portfolio created by the proposed approach than with the benchmark portfolio.

Table 5.5: Evaluation of the outranking relation between the solutions shown in Table 5.2. y: portfolio created by MOEA/D, x: portfolio created by the proposed approach

Comparison	Value	Outranks
$\beta(x, y)$	0.56	Yes
$\beta(y, x)$	0.0	No

We now perform the same analysis for every simulated DM and for every period in the dataset to have an idea of how satisfied the DM would be on average with the solutions provided by the proposed approach relative to the benchmark. The results are presented in Table 5.6. This table presents the proportion of times that the solutions of the proposed approach were better than the benchmark's solutions according to the

DM's actual system of preferences, $\mathscr{P}^i$, for the 20 simulated DMs, and for the 24 periods of the dataset. In each period, the solution created by our proposal for each DM is compared to each solution in the Pareto front approximated by MOEA/D. In this table, x represents the solutions provided by D_{sp} and y represents the solutions provided by MOEA/D. It is important to highlight that the actual simulated decision makers' systems of preferences are used in this assessment, as opposed to the directly elicited systems of preferences, which are the ones that the D_{sp} approach uses in its optimization procedure. The idea here is to assume that the actual investor has two possibilities: choosing one portfolio from the MOEA/D's approximation to the Pareto front, or choosing the portfolio built by the D_{sp} approach. What Table 5.6 shows is the average satisfaction (in terms of the interval-based outranking method) that the investor would have received if he/she would have chosen any option.

Table 5.6: Comparing the solutions x provided by the proposed approach and the solutions y provided by MOEA/D

Proportion of times that xSy is met	**Proportion of times that ySx is met**
0.1242	0.0904

The paired Wilcoxon test performed indicated that the difference of these means is considered to be statistically significant.

From Table 5.6 we can state that the proposed approach was able to find more satisfactory solutions than the benchmark algorithm when using a direct elicitation of the DM's system of preferences. Of course, this comparison is on the basis of the criteria contemplated in Problem (5.3.4). However, we can expect similar results of comparisons on the basis of alternative criteria and defer the validation of such hypothesis for future work. Finally, it is important to note here that the criteria contemplated in Problem (5.3.4) can be easily adjusted to the DM's specific requirements.

I_{sp} vs D_{sp} This subsection shows the comparison in performances of the portfolios created by our approach using both a direct elicitation of the DM's system of preferences, D_{sp}, and an indirect elicitation of such preferences, I_{sp}. The goal of the comparison is to define how "satisfied" the DM would be when applying the indirect elicitation procedure described in Chapter 4 and using the experimental design of Subsection 5.3.2.3 with respect to the direct elicitation described in Subsection 5.3.2.2. Thus, we calculate the proportion of times that one approach's solutions are preferred to the other approach's solutions. As stated in Sections 5.3.2.1 and 5.3.2.5, 20 instances are created in each experimental period and 24 periods are considered in the whole experimentation; thus, 480 experimental points are used to perform the comparison. Given that both approaches use the interval-based outranking method to represent the DM's preferences, this method is used to evaluate the DM's satisfaction. We use the actual system of preferences of the DM simulated in the ith instance, $\mathscr{P}^i$, to compare the solutions for $i = 1, \cdots, 20$.

As an example of the comparison, Table 5.7 shows the values in $\mathscr{P}^4$ and the corresponding values found by the elicitation procedures for the April 2014 period, Table 5.8 presents the portfolios created by the approaches for the elicited parameters, and Table 5.9 provides the fitness values of these portfolios in the sense of Problem (5.3.4).

Table 5.7: Decision policies of the simulated DM and parameter values found by the elicitation procedures in the April 2014 period

Decision policy	β_0	λ	w_1	w_2	w_3
Simulated DM	0.54	[0.52,0.57]	[0.57,0.84]	[0.31,0.38]	[0.00,0.06]
Direct elicitation	0.51	[0.65,0.66]	[0.50,1.00]	[0.25,0.46]	[0.00,0.04]
Indirect elicitation	0.51	[0.56,0.62]	[0.62,0.85]	[0.10,0.80]	[0.07,0.31]
Decision policy	w_4	v_1	v_2	v_3	v_4
Simulated DM	[0.02,0.12]	[0.04,0.05]	[0.09,0.10]	[0.33,0.42]	[0.11,0.13]
Direct elicitation	[0.02,0.10]	[0.03,0.04]	[0.08,0.11]	[0.26,0.47]	[0.09,0.09]
Indirect elicitation	[0.07,0.57]	[0.03,0.04]	[0.08,0.10]	[0.39,0.53]	[0.12,0.16]

Table 5.8: Portfolios built in the April 2014 period using the decision policies of Table 5.7

Stock	D_{sp}	I_{sp}
American Express Company (AXP)	0	0
Boeing Co. (BA)	0	0.018

Continued on next page

Table 5.8 – *Continued from previous page*

Stock	D_{sp}	I_{sp}
Caterpillar Inc. (CAT)	0	0
Cisco Systems, Inc. (CSCO)	0	0
Chevron Corporation (CVX)	0	0
EI du Pont de Nemours and Co (DWDP)	0.003	0
Walt Disney Company (DIS)	0	0
General Electric Company (GE)	0	0
Goldman Sachs Group Inc. (GS)	0	0
Home Depot, Inc. (HD)	0.05	0
International Business Machines Corporation (IBM)	0.28	0.347
Intel Corporation (INTC)	0	0
Johnson and Johnson (JNJ)	0.007	0
JPMorgan Chase and Co. (JPM)	0	0
Coca-Cola Company (KO)	0	0
McDonald's Corporation (MCD)	0.315	0.376
3M Co. (MMM)	0	0
Merck and Co., Inc. (MRK)	0	0
Microsoft Corporation (MSFT)	0	0
Nike Inc. (NKE)	0	0
Pfizer Inc. (PFE)	0	0.036
Procter and Gamble Co. (PG)	0	0
ATandT Inc. (T)	0.345	0.17
Travelers Companies Inc. (TRV)	0	0
UnitedHealth Group Inc. (UNH)	0	0
United Technologies Corporation (UTX)	0	0
Visa Inc. (V)	0	0.053
Verizon Communications Inc. (VZ)	0	0
Wal-Mart Stores Inc. (WMT)	0	0
Exxon Mobil Corporation (XOM)	0	0

Table 5.9: Fitness of the portfolios built by the approaches in Problem (5.3.4)

Criterion	D_{sp}	I_{sp}
70 percent confidence interval	[-0.0131, 0.0247]	[-0.0087, 0.0270]
99 percent confidence interval	[-0.0417, 0.0414]	[-0.0449, 0.0452]
Fundamental analysis' quality index	[0.3719,0.3720]	[0.3760,0.3761]
Technical analysis' quality index	[0.3533,0.3534]	[0.3393,0.3394]

Now, let y and x be the portfolios shown in Table 5.8, which were built using the D_{sp} and I_{sp}, respectively. According to the interval-based outranking method described in Subsection 2.3.4, the likelihood indexes that the DM whose preferences are $\mathscr{P}^4$ (shown in Table 5.7) assigns to these portfolios are shown in Table 5.10.

Table 5.10: Evaluation of the outranking relation between the solutions shown in Table 5.8. y: portfolio created using a direct elicitation, x: portfolio created using an indirect elicitation

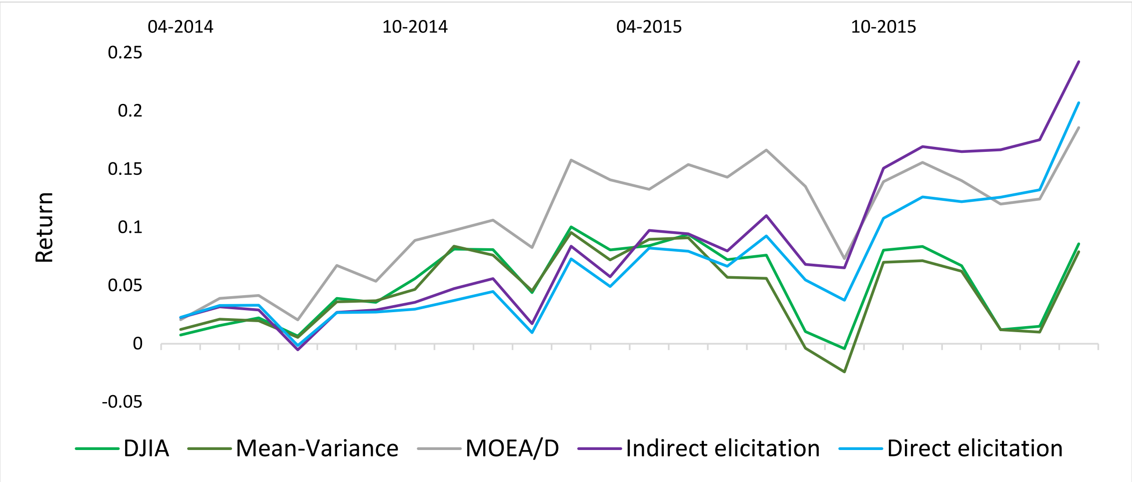
likelihood index of xSy	**Value**	**Strict preference**
$\beta(x, y)$	0.55	Yes
$\beta(y, x)$	0.45	No

Of course, there is not dominance between the portfolios but, according to Table 5.10, the solution found by the I_{sp} approach is strictly preferred to the D_{sp} approach's solution.

Applying the same analysis to the 480 points of comparison, we obtained that at least one of the portfolios found by the I_{sp} approach is strictly-preferred to 10% of the portfolios found by the D_{sp} approach, while at least one of the portfolios of the latter is strictly-preferred to 5% of the portfolios found by the former. When performing Student's t-test on difference of means with the null hypothesis that these proportions are the same, the two-tailed P value equals 0.0032. Hence, indicating that the difference is statistically significant with a 99% confidence level. This allows us to conclude that the indirect elicitation based on the preference disaggregation analysis proposed in Chapter 4 produced solutions that better satisfies the DM's preferences in the context of Problem (5.3.4) with respect to the direct elicitation procedure described in Subsection 5.3.2.2. Furthermore, these results together with the ones shown in Table 5.6, do not allow to reject hypothesis **H.1** of this document, which is "the proposed approach builds portfolios that are more preferred by the decision maker than the reference portfolios".

Performance in the objective space Now, we contrast the actual returns of the portfolios constructed by the proposed approach with those of three benchmarks: the Dow Jones Industrial Average (DJIA) index, the Mean-Variance approach [195], and the portfolios built using MOEA/D [295]. The problem addressed by the optimization methods (i.e., the Mean-Variance approach, MOEA/D and the approach proposed here) is provided in (5.3.4). The results are presented in Figure 5.1. This figure shows the accumulative monthly returns of the portfolios in the 24 periods span between April 2014 and March 2016. In the case of the Mean-Variance approach and MOEA/D, the accumulative return shown in Figure 5.1 corresponds to the average returns of the portfolios in their specific Pareto front. For the accumulative return of the DJIA index, we used the monthly returns published in Ref. [289]. In the case of our approach, the accumulative return is formed with the average returns of the portfolios obtained for the 20 DMs using first a direct elicitation procedure, D_{sp} (Subsection 5.3.2.2), and using later the indirect elicitation procedure of Chapter 4, I_{sp} (Subsection 5.3.2.3). Clearly, the proposed approach outperforms both the Mean-Variance approach and the DJIA index in these experiments no matter the elicitation procedure implemented.

Figure 5.1: Actual accumulative returns of the benchmark portfolios and the portfolios built by the proposed approach.



In other comparison, we can see similar average performances between MOEA/D and the proposed approach. Such similarity is originated in both approaches using the same criteria. However, the proposed approach generated the best accumulative return in the whole-time span; this indicates that the results with incorporation of DM's preferences have been, on average, more effective than the optimization without including preferences. It would be interesting to determine the DM's decision policies that generate the best returns. For example, a preliminary analysis suggests that when the simulated DMs fulfill a specific pattern in their systems of preferences the returns tend to grow. Some characteristics of the systems of preferences with such pattern are i) a greater importance assigned to the confidence intervals than to the financial indicators; ii) a greater importance assigned to the interval with the lower probability of containing the actual return; iii) a greater importance assigned to the fundamental analysis' quality index than to the technical analysis' one; and iv) higher relative values for the vetoes assigned to the financial indicators. Given that the main goal of this Section is to provide an illustrative example of the proposed methodology (Subsection 5.2), we defer the analysis of the previous and similar assertions for future work.

In the context of the comparison of MOEA/D with our approach for a given period of time, it is important to highlight that the performance of the first is the average return of the portfolios in its Pareto front approximation. Thus, the performance of MOEA/D shown in Figure 5.1 could be seen as the average performance of several attitudes facing risk, while the performance of our approach is the only portfolio representing a highly risk-averse investor. It is plausible to assume that if the market presents an uptrend (as slightly seen in the considered time span) then the return of the solutions created by MOEA/D should be, on average, better to the solution of our approach.

Finally, the proposed approach has good behavior in the presence of losses, specifically in the periods of August and September 2015 where the steepest fall of the market occurred. This indicates us a satisfactory protection against risk. Moreover, the approach is taking evident advantage of the market upturns, what indicates that it is also capable of finding uptrend opportunities.

5.3.3.2 Assessing the satisfaction of the decision maker

Here, we evaluate our approach in its ability to define a set of preference parameters through which

i. it is possible to establish the same preference relations as the ones inferred from the assignments made by the simulated DMs (see Assumptions (4.2.1) to (4.2.6)); and

ii. it is possible to suggest the same assignments as the ones made by the simulated DMs when creating the reference sets (cf. Subsection 5.3.2.3).

Using preference relations as a benchmark In order to test the proposed approach's effectiveness to reproduce the same binary preference relations as the simulated DMs, we use Equation (4.4.9), which is a measure based on the proportion of times that the preference relations coincide. We use the DM's actual decision model in each of the 20 instances for each of the 24 experimental periods to create a new set of decisions, χ. In this new set, the DM assigns 100 new portfolios to three categories, $C_3 = Good$, $C_2 = Doubt$ and $C_1 = Bad$. The effectiveness of the proposed approach to reproduce the preference relations inferred from this classification is defined for each instance and each experimental period as (cf. Subsection 4.4.2):

$$1 - \frac{\xi}{\eta}.$$

Where

$\eta = \frac{TN}{2}(TN - 1)$,

$TN = |\chi| = 100$,

$\xi = \sum_{(x,y)\in\chi\times\chi} [\xi_P(x, y) + \xi_Q(x, y) + \xi_K(x, y) + \xi_I(x, y) + \xi_R(x, y)]$,

$\xi_P(x, y) = 1$ if $xP_{\mathscr{P}'}y \implies C_P(x) > C_P(y)$ is false and 0 otherwise,

$\xi_Q(x, y) = 1$ if $xQ_{\mathscr{P}'}y \implies C_P(x) \geq C_P(y)$ is false and 0 otherwise,

$\xi_K(x, y) = 1$ if $xK_{\mathscr{P}'}y \implies C_P(x) \geq C_P(y)$ is false and 0 otherwise,

$\xi_I(x, y) = 1$ if $xI_{\mathscr{P}'}y \Rightarrow C_P(x) = C_P(y)$ is false and 0 otherwise, and

$\xi_R(x, y) = 1$ if xOy; O indicates that the relation between x and y is other than the ones defined in Subsection 2.3.4.

Using this measure, the effectiveness of our approach using the direct elicitation described in 5.3.2.2 in the 480 experimental points is 84%. While the average effectiveness of our approach using the indirect elicitation proposed in Chapter 4 is 96%. The difference is statistically significant according to student's t test. Indicating that the latter approach was able to better reproduce the DM's preferences in the context of binary preference relations.

Using ordinal classification as a benchmark Now, the solutions found by our approach implementing both the direct and indirect elicitation are used to assign portfolios in preferentially ordered categories. So, the quality of the solutions is revised by contrasting the assignments made by our approach and by the actual simulated DMs. The comparison is based on the new sets of decisions defined in the previous subsection. All the assignments are performed as described in Subsection 5.3.2.3. Our approach's effectiveness is defined as the proportion of portfolios that were assigned to the same category as the DMs' assignments in the 480 experimental points.

The effectiveness in the context of ordinal classification for our approach using a direct elicitation is 70%; whereas its effectiveness using indirect elicitation is 80%. This difference is statistically significant according to student's t test. Indicating that the indirect elicitation procedure proposed in Chapter 4 is more effective in reproducing the DM's assignments.

Chapter 6

Conclusions and future work

In this doctoral thesis, we discussed the Portfolio Optimization Problem and how the preferences of the investor as well as his/her attitude facing risk should be incorporated during the search of the *best* portfolio. This is a crucial stage in the broader topic of Portfolio Selection, a highly important financial activity in today's world economy. Such topic is so important that, for example, the stock Portfolio Selection is considered by some authors as "crucial to the existence of capitalism and private property" [241].

We highlighted in this work the importance of considering the case where the investor's preferences are ambiguous, vague, ill-determined or imprecise. Thus, our objective was to propose and validate an approach that addresses the Portfolio Optimization Problem finding solutions that are most preferred by the investor in presence of imperfect knowledge.

Focusing on the most relevant characteristics that preference and Portfolio Selection models must fulfill according to the related literature, we assumed that the approach needed to be able to: i) manage risk in the objectives whose impacts cannot be exactly known in such a way as to incorporate the decision maker's attitude when facing this risk, ii) model the decision maker's preferences so his/her holistic decisions can be reproduced, and iii) deal with many criteria. Let us now address these lines of thought in form of the thesis' research questions.

How to manage risk in the objectives whose impacts generated by the portfolios cannot be exactly known, in such a way as to incorporate the decision maker's attitude during Portfolio Optimization? We described in Chapter 3 our idea to use confidence intervals as underlying criteria to risky objectives. With the purpose of assessing this idea, we showed an illustrative example where the most common risky objective in literature was considered, maximization of the portfolio return. To select portfolios on the basis of confidence intervals, the optimization procedure was performed using the so-called Interval Analysis Theory (Section 2.2). Accordingly, we enhanced in Section 3.3 a widely accepted multiobjective evolutionary algorithm based on decomposition, MOEA/D, to deal with objective functions defined as *interval numbers*. Furthermore, we implemented some improvements to increase the diversity of solutions provided by the evolutionary algorithm.

An extensive validation of the proposal was performed, where out-of-sample historical data from the stocks in the Dow Jones Industrial Average index (Subsection 2.1.4.3) was used to perform a back-testing strategy. The proposal was compared in 156 scenarios against the index, the classical and risk aversion formulations of the Mean-Variance optimization (Subsection 2.1.1), and a recently published work [124]. We used two confidence intervals in the optimization process as criteria to represent the investor's behavior facing risk. And two different behaviors were simulated. First, a highly risk-averse investor, and later a lowly risk-averse investor. The optimization problem for assessing the proposal in this illustrative example was formalized in Section 3.2.

The results shown in Figures 3.3-3.10 allowed us to conclude that the proposal was effective in the construction of portfolios when the objective is maximization of return. Our proposal outperformed all the benchmarks in most of the 156 scenarios, giving a considerably better accumulated return after an optimization with 13 years of historical data. These results are strong out-of-sample evidence that confidence intervals provide useful characterizations of the portfolios' returns and their involved risk. Furthermore, we confirm with these results that an active management can in fact be achieved and that greater advances should be sought in this stream of thought.

Comparable with the importance of the previous results, the analysis performed in Subsection 3.4.4 allowed us to see that the proposal presented robustness in the more critical period of the last years, specifically March 2008-February 2009. This is a crucial result in the experiments because the behaviors of the investors modeled are risk-averse. Our results confirm that i) our proposal was adept to find portfolios by explicitly considering the DM's attitude facing risk, being conservative when the DM's behavior was highly risk averse and taking good advantage of the uptrends when the behavior was lowly risk averse; and ii) confidence intervals were a useful risk measure in the 2008 crisis, since they helped to reduce losses in the period.

How to model the decision maker's preferences in such a way that the proposed approach can reproduce his/her holistic decisions? In Chapter 4 we considered that the values of the parameters in the decision maker's system of preferences are imperfectly known. Furthermore, we assumed that imperfect knowledge about such preferences can be encompassed as interval numbers. Thus, we proposed a Preference Disaggregation Analysis method based on Interval Theory model to indirectly elicit the decision maker's preference parameters. We extensively assessed the proposed model in several reference sets of assignments made by simulated decision makers. The experiments were based on two ways to measure the indirect elicitation's effectiveness: comparing coincidences of binary preference relations and coincidences of assignments to ordered categories.

The results shown in Tables 4.3, 4.6, 4.7 and 4.8 support that the effectiveness of our proposal is high -in most cases superior to 99%- when the portfolios are described by six and twelve criteria. This effectiveness is measured as the average proportion of coincidences of the binary preference relation inferred from the assignments made by the simulated DM and the one inferred from the elicited decision model, for 40 instances. On the other hand, Table 4.9 shows that when we obtain the average number of coincidences between the assignments made by the simulated DM and the ones suggested by the elicited decision model, for 40 instances, the effectiveness of the method is also appreciably high. We found that, generally, the effectiveness of the proposed approach is greater as the cardinality of the set of reference DM's decisions increases.

How to select the portfolios that are more preferred by the decision maker when many criteria are considered? We presented in Chapter 5 our approach to select the most preferred portfolio managing uncertainty and incorporating the decision maker's preferences. Such approach allows the investor to specify as many criteria as she/he requires by aggregating all the criteria, through Fuzzy Logic and the investor's preferences, in a bi-criterion optimization problem. Addressing this underlying problem instead of the original one allows the approach to perform a guided search towards the most preferred solutions within the Pareto front, thus reducing the investor's difficulty to select the final portfolio. Finally, the uncertainty involved in the investor's preferences as well as in the actual return of the portfolio are both encompassed as ranges of numbers through the so-called Interval Theory.

An illustrative use case was provided in the context of the stock Portfolio Optimization Problem. The proposed approach was assessed considering i) the investor's own system of preferences, and ii) the actual returns of the portfolios provided. A comparison with some well-known benchmarks, the Dow Jones Industrial Average index, the Mean-Variance method, and an approach based on MOEA/D, was performed. We conclude from Tables 5.5 and 5.6 that, for the illustrative example, the proposed approach was able to deal with many criteria and, at the same time, find solutions that satisfy the investor's preferences better than the solutions provided by the corresponding benchmark. From Figure 5.1, we conclude that the proposed approach was able to outperform the benchmarks in the objective space; that is, it found portfolios with greater actual returns. It can also be seen from this figure that our proposal suffered less aggressive falls than the benchmarks (showing good management of the involved risk) and that it had better exploitation of the rises (showing a good identification of opportunities).

The above remarks indicate us that the proposed approach i) allows the DM to deal with as many criteria as she/he considers necessary to satisfy her/his requirements of information (of course, the method allows the hypothesis used in the illustrative example to be specifically suited to the actual DM's necessities); and ii) uses appropriate elements to construct portfolios that maximize the impact on both the underlying criteria and the final objective in risky and non-risky environments.

We also assessed our approach with respect to the method used to elicit the decision maker's system of preferences. Particularly, we compared the performance of the approach when using the direct elicitation procedure described in Subsection 5.3.2.2 with the indirect elicitation procedure implementing the interval-based preference disaggregation analysis proposed in Chapter 4. It is important to note that, differently to the experiments performed in Section 4.4, the experiments performed in Section 5.3.2 and referenced below were implemented using real data from the DJIA index. We evaluated and contrasted the out-of-sample effectiveness of our approach in both scenarios and conclude that its performance is significantly better using the indirect elicitation proposed in Chapter 4 for all scenarios. These scenarios are related to i) the decision maker's satisfaction, ii) the approach's ability to reproduce the decision maker's decisions in the context of binary preference relations, and iii) the approach's ability to reproduce the decision maker's decisions in the context of ordinal classification. Hence, we have found evidence that support the hypothesis exposed by Mousseau and Slowinski in Ref. [216] about the superiority of indirect elicitation procedures. Finally, the statistical results presented in Section 5.3.3.1 (particularly those obtained from Tables 5.10 and 5.6) do not allow to reject the hypothesis that "the proposed approach builds portfolios that are more preferred by the decision maker than the reference portfolios". Therefore, indicating that the proposals presented in this paper can be of great help to real-world investors and, by helping real-world investors, the proposals of this thesis can provide relevant impacts and advances of the global economy.

We emphasize that although the main proposal was applied in the context of resource allocation, there is a wide range of problems where the proposal can be applied. The general characteristics of such problems are: i) requirement of the preferences of a decision maker to make the final decision, ii) consideration of many criteria, iii) uncertainty in the preferences of the decision maker and/or the impact of the solution alternatives on the criteria.

Future work When considering the results of our proposal to manage risk, we saw that more actual portfolio returns than expected fell outside their respective confidence intervals, thus reducing the performance of the approach. This effect might be mitigated when confidence intervals are around alternative estimators other than the expected return, or when a more precise analysis of the probability within the intervals is made. Thus, a validation of the approach using different types of estimators of return, their combination and a method to scrutinize the probability within the intervals is deferred as future work.

One future interesting research line is to assess new ways to generate the reference sets; for example using alternative sorting methods, different cardinalities of the reference sets and the criteria sets, and/or diverse number of categories. Another research line consists in modifying Assumption 5 in Section 4.2 (specifically Equations (4.2.1)-(4.2.6)) to introduce new preference relations and/or increase the granularity of the system of relations.

Other future lines of work are i) the analysis of the specific characteristics of the investor's system of preferences that allow to obtain greater returns; ii) the assessment of the proposed approach in different contexts with respect to the number and/or kind of criteria, the nature of investment objects within the portfolios, and the number of elements in the investment objects universe; iii) the consideration of specific useful characteristics of the portfolio problem, such as multiple period optimization and a higher number of constraints; iv) a more elaborated model using fuzzy logic to handle partial truth, specially in Equations (5.3.2) and (5.3.3); v) to perform an interactive procedure where the DM gets to know the shape of the Pareto front's geometrical characteristics.

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Appendix A

Comparing the built preferences and the actual DM's system of preferences

Table A.1: Comparison of the cutting level β_0 of the simulated DM and the one found by the method in χ_{30}

Instance	β_0	
	DM	**PDA**
1	0.515	0.529
2	0.541	0.547
3	0.516	0.576
4	0.549	0.56
5	0.541	0.547
6	0.544	0.682
7	0.549	0.562
8	0.513	0.518
9	0.536	0.555
10	0.519	0.526

Table A.2: Comparison of the cutting level λ of the simulated DM and the one found by the method in χ_{30}

Instance	λ	
	DM	**PDA**
1	[0.529,0.541]	[0.559,0.67]
2	[0.512,0.523]	[0.56,0.667]
3	[0.522,0.54]	[0.567,0.67]
4	[0.515,0.523]	[0.561,0.668]
5	[0.52,0.527]	[0.553,0.659]
6	[0.521,0.534]	[0.557,0.666]
7	[0.516,0.529]	[0.557,0.661]
8	[0.522,0.536]	[0.561,0.666]
9	[0.529,0.541]	[0.571,0.676]
10	[0.518,0.54]	[0.566,0.641]

Table A.3: Comparison of the vector of weights of the simulated DM and the one found by the method in χ_{30}

Instance	Decision policy	w_1	w_2	w_3	w_4	w_5	w_6
1	DM	[0.102,0.231]	[0.125,0.208]	[0.104,0.23]	[0.131,0.202]	[0.103,0.231]	[0.117,0.216]
	PDA	[0.115,0.218]	[0.118,0.215]	[0.114,0.219]	[0.117,0.216]	[0.117,0.216]	[0.119,0.215]
2	DM	[0.086,0.247]	[0.11,0.224]	[0.118,0.215]	[0.112,0.221]	[0.098,0.236]	[0.127,0.207]
	PDA	[0.122,0.212]	[0.116,0.217]	[0.116,0.217]	[0.118,0.215]	[0.121,0.212]	[0.112,0.221]
3	DM	[0.133,0.201]	[0.115,0.219]	[0.11,0.223]	[0.14,0.193]	[0.144,0.189]	[0.091,0.242]
	PDA	[0.118,0.215]	[0.118,0.215]	[0.116,0.218]	[0.115,0.219]	[0.118,0.215]	[0.115,0.218]
4	DM	[0.093,0.241]	[0.126,0.207]	[0.135,0.198]	[0.112,0.221]	[0.098,0.235]	[0.12,0.213]
	PDA	[0.118,0.215]	[0.118,0.215]	[0.112,0.221]	[0.118,0.215]	[0.124,0.21]	[0.114,0.219]
5	DM	[0.114,0.219]	[0.088,0.245]	[0.133,0.2]	[0.119,0.214]	[0.11,0.223]	[0.136,0.197]
	PDA	[0.12,0.214]	[0.119,0.215]	[0.113,0.22]	[0.115,0.218]	[0.117,0.217]	[0.119,0.214]
6	DM	[0.091,0.242]	[0.144,0.19]	[0.121,0.212]	[0.116,0.217]	[0.149,0.184]	[0.131,0.202]
	PDA	[0.117,0.216]	[0.114,0.219]	[0.115,0.219]	[0.117,0.216]	[0.116,0.217]	[0.116,0.217]
7	DM	[0.106,0.227]	[0.135,0.198]	[0.107,0.226]	[0.119,0.215]	[0.137,0.197]	[0.092,0.242]
	PDA	[0.115,0.218]	[0.116,0.217]	[0.117,0.217]	[0.12,0.214]	[0.119,0.214]	[0.114,0.219]
8	DM	[0.129,0.204]	[0.095,0.239]	[0.104,0.229]	[0.122,0.211]	[0.091,0.242]	[0.084,0.25]
	PDA	[0.121,0.212]	[0.119,0.214]	[0.116,0.217]	[0.119,0.215]	[0.112,0.221]	[0.117,0.217]
9	DM	[0.112,0.221]	[0.103,0.23]	[0.116,0.217]	[0.107,0.227]	[0.12,0.213]	[0.131,0.202]
	PDA	[0.115,0.218]	[0.118,0.216]	[0.119,0.214]	[0.12,0.213]	[0.12,0.213]	[0.118,0.215]
10	DM	[0.138,0.196]	[0.125,0.209]	[0.095,0.238]	[0.122,0.211]	[0.123,0.21]	[0.101,0.232]
	PDA	[0.118,0.216]	[0.117,0.216]	[0.115,0.218]	[0.12,0.213]	[0.113,0.221]	[0.118,0.216]

Table A.4: Comparison of the vector of vetoes of the simulated DM and the one found by the method in χ_{30}

Instance	Decision policy	w_1	w_2	w_3	w_4	w_5	w_6
1	DM	[0.652,0.807]	[0.663,0.842]	[0.657,0.82]	[0.604,0.703]	[0.753,0.871]	[0.757,0.864]
	PDA	[0.663,0.795]	[0.635,0.745]	[0.733,0.904]	[0.719,0.863]	[0.733,0.902]	[0.733,0.915]
2	DM	[0.612,0.697]	[0.616,0.786]	[0.64,0.747]	[0.698,0.812]	[0.746,0.895]	[0.755,0.835]
	PDA	[0.665,0.797]	[0.718,0.875]	[0.651,0.78]	[0.703,0.849]	[0.629,0.74]	[0.735,0.9]
3	DM	[0.685,0.82]	[0.686,0.793]	[0.699,0.875]	[0.649,0.838]	[0.635,0.762]	[0.616,0.761]
	PDA	[0.736,0.911]	[0.676,0.807]	[0.677,0.807]	[0.682,0.816]	[0.629,0.746]	[0.653,0.788]
4	DM	[0.663,0.841]	[0.678,0.783]	[0.706,0.821]	[0.701,0.9]	[0.715,0.873]	[0.732,0.873]
	PDA	[0.7,0.849]	[0.713,0.854]	[0.697,0.836]	[0.714,0.873]	[0.698,0.841]	[0.735,0.897]
5	DM	[0.688,0.833]	[0.647,0.808]	[0.668,0.834]	[0.662,0.794]	[0.649,0.83]	[0.656,0.781]
	PDA	[0.684,0.849]	[0.692,0.828]	[0.661,0.802]	[0.734,0.905]	[0.648,0.772]	[0.707,0.863]
6	DM	[0.75,0.936]	[0.661,0.842]	[0.696,0.794]	[0.618,0.787]	[0.713,0.787]	[0.705,0.818]
	PDA	[0.669,0.796]	[0.723,0.889]	[0.706,0.843]	[0.667,0.799]	[0.707,0.854]	[0.633,0.746]
7	DM	[0.655,0.758]	[0.709,0.911]	[0.747,0.854]	[0.639,0.782]	[0.713,0.877]	[0.76,0.838]
	PDA	[0.652,0.774]	[0.706,0.868]	[0.644,0.757]	[0.705,0.859]	[0.737,0.904]	[0.727,0.893]
8	DM	[0.674,0.775]	[0.731,0.866]	[0.61,0.771]	[0.746,0.832]	[0.714,0.829]	[0.689,0.858]
	PDA	[0.689,0.815]	[0.73,0.907]	[0.715,0.866]	[0.733,0.899]	[0.715,0.874]	[0.654,0.784]
9	DM	[0.617,0.736]	[0.626,0.778]	[0.685,0.771]	[0.673,0.812]	[0.725,0.817]	[0.707,0.853]
	PDA	[0.637,0.757]	[0.685,0.82]	[0.728,0.862]	[0.693,0.829]	[0.646,0.771]	[0.648,0.77]
10	DM	[0.622,0.724]	[0.725,0.867]	[0.64,0.813]	[0.713,0.897]	[0.654,0.75]	[0.616,0.689]
	PDA	[0.622,0.742]	[0.674,0.811]	[0.696,0.833]	[0.679,0.816]	[0.719,0.877]	[0.644,0.763]

Appendix B

Benchmarks used in this thesis

Table B.1: Characteristics of the benchmarks used

Benchmark	Characteristics	Thesis pages
Dow Jones Industrial Average index (DJIA)	Commonly used financial benchmark.	33, 76, 81, 127
Mean-variance model with bounds on individual stocks constraint (MV)	Refinement of the classical approach in Modern Portfolio Theory	13, 80, 127
Dataset from recent literature benchmark (RB)	Results provided by Ref. [124]. Its dataset is a subset of the one used in this thesis.	80 - 81
MOEA/D	Most mentioned MOEA based on decomposition of the literature	53, 119

Table B.2: Procedures used to assess the proposed approach's effectiveness

Procedure	Benchmarks	Description	Pages
Assessing the uncertainty management proposal	DJIA, MV and RB	Estimates the approach's ability to consider the investor's attitude in presence of risk. It consists in using confidence intervals as criteria underlying the maximization of portfolio return, a risky objective.	72-86
Assessing the proposed elicitation model in the context of preference relations		Evaluates the approach's effectiveness to reproduce the investor's preferences by comparing binary relations inferred from his/her holistic decisions and the ones inferred from the decision models built by the approach.	97-101
Assessing the proposed elicitation model in the context of ordinal classification		Evaluates the approach's effectiveness to reproduce the investor's decisions by comparing the investor's and the approach's assignments of portfolios to ordered categories	101-102
Assessing the general performance of the approach and the satisfaction of the investor in presence of many criteria	DJIA, MV and MOEA/D	All the previous characteristics are taken into consideration to address the Portfolio Optimization Problem when the investor is interested in many criteria. The approach's performance is assessed in function of the impact on the underlying criteria and the maximization of the portfolio return. The investor's satisfaction is obtained as the proportion of coincidences between the approach's recommendations and the investor's holistic decisions.	129-130